HARMONIC MAPPINGS BETWEEN RIEMANNIAN MANIFOLDS

by

Anand Arvind Joshi

A Thesis Presented to the
FACULTY OF THE GRADUATE SCHOOL
UNIVERSITY OF SOUTHERN CALIFORNIA
In Partial Fulfillment of the
Requirements for the Degree
MASTER OF ARTS
(MATHEMATICS)

August 2006

Dedication

I dedicate this thesis to my parents Arvind and Suvarna Joshi.

Table of Contents

Abstract

Harmonic mappings between two Riemannian manifolds is an object of extensive study, due to their wide applications in mathematics, science and engineering. Proving the existence of such mappings is challenging because of the non-linear nature of the corresponding partial differential equations. This thesis is an exposition of a theorem by Eells and Sampson, which states that any given map from a Riemannian manifold to a Riemannian manifold with non-positive sectional curvature can be freely homotoped to a harmonic map. In particular, this proves the existence of harmonic maps between such manifolds. The technique used for the proof is the heat-flow method.

Chapter 1

Harmonic Mappings

In this chapter we define and discuss harmonic mappings. Let (M, g) and (N, h) be m and n dimensional Riemannian manifolds, and let u denote a smooth map from M to N, i.e. $u \in C^\infty(M, N)$. A natural question to ask is: what is the 'least expanding' map from M to N? In order to make precise what we mean by 'least expanding' map here, we need to analyze the space of maps $C^\infty(M, N)$. In the sections that follow, we do this analysis and define an *energy of maps* on this space. A harmonic map will be a critical point of this energy as discussed later.

1.1 Space of Maps

Let T_xM denote the tangent space of M and let T_xM^* be the dual space of this tangent space. We know that $u \in C^\infty(M, N)$ implies that du_x is a linear map from T_xM to $T_{u(x)}N$, i.e. $du_x \in \mathrm{Hom}(T_xM, T_{u(x)}N)$. We want to find a metric on $\mathrm{Hom}(T_xM, T_{u(x)}N)$ so that we can define energy of maps. We first prove a lemma.

Lemma 1.1.1. $T_xM \cong T_xM^*$. *The isomorphism is linear and induces a metric on* T_xM^*.

Proof. The Riemannian metric g induces a natural linear isomorphism between T_xM and its dual T_xM^* defined as follows. Let $X_x = \sum_{i=0}^m X^i(x)(\frac{\partial}{\partial x^i})_x \in T_xM$ and $w_x = \sum_{i=0}^m w_i(x)(dx^i)_x \in T_xM^*$. Define $\flat : T_xM \to T_xM$ and $\sharp : T_xM^* \to T_xM$

$$X_x^\flat = \sum_{i=1}^m (\sum_{i=1}^m g_{ij}(x)X^j(x))(dx^i)_x, \tag{1.1}$$

$$w^\sharp = \sum_{i=1}^m (\sum_{i=1}^m g^{ij}(x)w_j(x))(\frac{\partial}{\partial x^i})_x. \tag{1.2}$$

Clearly $\sharp$ and $\flat$ are linear. Also it can be verified that they are inverse of each other resulting in a linear isomorphism between T_xM and T_xM^*. Now we define a metric g_x^* on T_xM^* by

$$g_x^*(w_x, \theta_x) = g_x(w_x^\sharp, \theta_x^\sharp) \quad \text{for} \quad w_x, \theta_x \in T_xM^*. \tag{1.3}$$

This bilinear form g_x^* is a metric due to linearity of $\sharp$. We can also get $g_x^*((d_x^i)_x, (d_x^j)_x) = g_x^{ij}$ where (g^{ij}) denotes the matrix inverse of $g = (g_{ij})$. $\square$

Proposition 1.1.2. $\mathrm{Hom}(T_xM, T_{u(x)}N) \cong T_xM^* \otimes T_{u(x)}N$.

Proof. For every $f \in \mathrm{Hom}(T_xM, T_{u(x)}N)$ we associate a bilinear map $f^\dagger \in T_xM^* \otimes T_{u(x)}N$ by defining

$$f^\dagger(V, w) = w(f(V)), \quad \forall V \in T_xM, w \in T_{u(x)}N^*. \tag{1.4}$$

We can see that, given such a bilinear map, we can also associate with it a linear map in $\mathrm{Hom}(T_xM, T_u(x))$. $\square$

We know that $du_x \in \mathrm{Hom}(T_xM, T_{u(x)}N)$ is represented in local coordinates by

$$du_x\left(\left(\frac{\partial}{\partial x^i}\right)\right) = \sum_{\alpha=1}^m \left(\frac{\partial u^\alpha}{\partial x^i}\right)(x)\left(\frac{\partial}{\partial y^\alpha}\right)_{u(x)}. \tag{1.5}$$

Since the basis for $T_xM^* \otimes T_{u(x)}N$ is given by

$$(dx^i)_x \otimes (\frac{\partial}{\partial y^\alpha})_{u(x)}, \tag{1.6}$$

du_x is represented by

$$du_x = \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \left(\frac{\partial u^\alpha}{\partial x^i}\right)(x)(dx^i)_x \otimes \left(\frac{\partial}{\partial y^\alpha}\right)_{u(x)}. \tag{1.7}$$

We proved that g^{ij} is an inner product on T_xM^*. Also $h_{u(x)}$ is the induced inner product in $T_{u(x)}$. These two inner products induce an inner product on $T_xM^* \otimes T_u(x)N$ given by

$$\left\langle (dx^i)_x \otimes \left(\frac{\partial}{\partial y^\alpha}\right)_{u(x)}, (dx^j)_x \otimes \left(\frac{\partial}{\partial y^\beta}\right)_{u(x)} \right\rangle = g^{ij} h_{\alpha\beta}(u(x)). \tag{1.8}$$

Since this inner product is defined everywhere on M, we define an inner product on sections by setting

$$\langle \sigma, \sigma' \rangle(x) = \langle \sigma(x), \sigma'(x) \rangle_x, \quad \text{for} \quad x \in M; \sigma, \sigma' \in \Gamma(T^*M \otimes u^{-1}TN). \tag{1.9}$$

With this inner product, we define a norm on du_x given by

$$|du|^2 = \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij} h_{\alpha\beta}(u) \left(\frac{\partial u^\alpha}{\partial x^i}\right)\left(\frac{\partial u^\beta}{\partial x^j}\right). \tag{1.10}$$

With this norm defined on $\mathrm{Hom}(TM, TN)$, we now define the energy density of a map.

Definition 1.1.1. Given $u \in C^\infty(M, N)$, the *energy density function* of u is defined as

$$e(u)(x) = \frac{1}{2} |du|^2 (x), \quad x \in M. \tag{1.11}$$

Definition 1.1.2. Let (M, g) be a compact Riemannian manifold. Given $u \in C^\infty(M, N)$, the *energy* or *harmonic energy* of u is defined as

$$E(u) = \int_M e(u) d\mu_g = \int_M \frac{1}{2} |d\mu_g|. \tag{1.12}$$

The *energy density* of u can be interpreted in a following way. Let $\{e_1, ..., e_m\}$, and $\{e'_1, ..., e'_n\}$ be orthonormal bases with respect to g_x and $h_{u(x)}$, for tangent spaces $T_x M$ and $T_{u(x)} N$, respectively. We express du_x in these bases as

$$du_x(e_i) = \sum_{\alpha=1}^{n} \lambda_i^\alpha e'_\alpha, \quad i = 1, ..., m. \tag{1.13}$$

Then, we get

$$|du|^2(x) = \sum_{i=1}^{m} \sum_{\alpha=1}^{n} (\lambda_i^\alpha)^2. \tag{1.14}$$

Consequently, we can regard the energy density functional $e(u)(x)$ as the 'rate of expansion' of the differential $du_x : T_x M \to T_{u(x)} N$ of u at $x \in M$. This is why we call $e(u_x)$ 'energy density' of the map.

Thus the energy $E(u)$ is defined for each $u \in C^\infty(M, N)$. The energy of maps E can be regarded as a functional $E : C^\infty(M, N) \to \mathbb{R}$, and we want to find maps which are critical points of this functional E.

1.2 Connections in the Space of Maps

Having introduced an inner product and norm for $u \in \mathrm{Hom}(M, N)$ i.e. on $\Gamma(TM^* \otimes u^{-1}TN)$ we want to know what is the effect on energy if the map u is changed by a small amount. In other words, we want to be able to take directional derivatives in the space $\Gamma(TM^* \otimes u^{-1}TN)$. To do that, we develop the notion of connection for this space. First, we develop connections for TM^* and $u^{-1}TN$.

Let ∇ denote the Levi-Civita connection of M which gives us a map

$$\nabla : \Gamma(TM) \to \Gamma(TM^* \otimes TN) \tag{1.15}$$

which assigns a tensor field $\nabla Y \in \Gamma(TM^* \otimes TM)$ of type $(1,1)$ to $Y \in \Gamma(TM)$, a $(0,1)$ tensor field. ∇Y is the *covariant differential* of Y. Let $\sharp$ and $\flat$ be the isomorphisms between TM and TM^* given in (1.1) and (1.2). We can define a connection ∇^* in TM^* by setting

$$\begin{aligned}
\nabla^*_X w(Y) &= (\nabla_X w^\sharp)^\flat(Y), \quad Y \in \Gamma(TM), w \in \Gamma(TM^*) && (1.16)\\
&= g_x(\nabla_X w^\sharp, Y) && (1.17)\\
&= X(g_x(w^\sharp, Y)) - g_x(w^\sharp, \nabla_X Y) \quad \text{by compatibility of } \nabla && (1.18)\\
&= Xw(Y) - w(\nabla_X Y) && (1.19)
\end{aligned}$$

which could also serve as an alternate definition for the connection ∇^* and also explains that the connection ∇^* on TM^* and connection ∇ on TM can be regarded as dual to each other.

Because of the compatibility of ∇ with g_{ij}, we can see that the connection ∇^* is compatible with the metric g^{ij} on TM^* by the following computation.

Lemma 1.2.1. $Xg^*(w,\theta) = g^*(\nabla^*_X w, \theta) + g^*(w, \nabla^*_X, \theta)$.

Proof.

$$RHS = g((\nabla^*_X w)^\sharp, \theta^\sharp) + g(w^\sharp, (\nabla^*_X \theta)^\sharp) \tag{1.20}$$
$$= g(\nabla_X w^\sharp, \theta^\sharp) + g(w^\sharp, \nabla_X \theta^\sharp) \tag{1.21}$$
$$= Xw(\theta^\sharp) = LHS \tag{1.22}$$

□

Thus ∇^* is a Riemannian connection. Now that we have introduced the connection ∇^* in TM^*, what are the connection coefficients? We use (1.19) to do the computation.

$$(\nabla^*_{\frac{\partial}{\partial x^j}} dx^k)(\frac{\partial}{\partial x^l}) = \frac{\partial}{\partial x^j} dx^k(\frac{\partial}{\partial x^l}) - dx^k(\nabla_{(\frac{\partial}{\partial x^j})}\frac{\partial}{\partial x^l}) \tag{1.23}$$
$$= \frac{\partial}{\partial x^j}\delta^k_l - \Gamma^k_{jl} \tag{1.24}$$
$$= -\Gamma^k_{jl} \tag{1.25}$$

Thus we get an expression for ∇^* as

$$\nabla^*_{\frac{\partial}{\partial x^i}} dx^k = -\sum_{j=1}^{m} \Gamma^k_{ij} dx^j, \quad 1 \leq i, k \leq m. \tag{1.26}$$

We note that the connection coefficients of ∇^* induced in TM^* from ∇ are negative of the connection coefficients of ∇.

Now consider the tangent bundle $u^{-1}TN \subset TM$ induced from TN by the map $u : M \to N$. At each point x,

$$\left\{ \left(\frac{\partial}{\partial y^1} \circ u \right)(x), ..., \left(\frac{\partial}{\partial y^n} \circ u \right)(x) \right\} \tag{1.27}$$

gives rise to a base for the fiber $T_{u(x)}N$ of $u^{-1}TN$ over x. We introduce a connection $'\nabla$ in $u^{-1}TM$ from the connection ∇' in TN by defining

$$\left('\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial y^\gamma}\circ u\right)(x)=\nabla'_{du_x\left(\frac{\partial}{\partial x^i}\right)_x}\frac{\partial}{\partial y^\gamma}. \tag{1.28}$$

If $\Gamma'^{\alpha}_{\beta\gamma}$ are the connection coefficients for the connection ∇' on TN, then from (1.27) and (1.28), we get

$$\left('\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial y^\gamma}\circ u\right)(x)=\sum_{\alpha=1}^{n}\left(\sum_{\beta=1}^{n}\frac{\partial u^\beta}{\partial x^i}(x)\Gamma'^{\alpha}_{\beta\gamma}(u(x))\right)\left(\frac{\partial}{\partial y^\alpha}\circ u\right) \tag{1.29}$$

namely, $'\nabla$ is a linear connection in $u^{-1}TN$ with the connection coefficients given by

$$\left\{\sum_{\beta=1}^{n}\frac{\partial u^\beta}{\partial x^i}\Gamma'^{\alpha}_{\beta\gamma}(u)|1\leq i\leq m, 1\leq\alpha,\gamma\leq n\right\}. \tag{1.30}$$

We call this $'\nabla$ the *induced connection* in $u^{-1}TN$. Also if $h\in\Gamma(TN\otimes TN)$ is a metric in TN, then $u^*h=h_{u(x)_{x\in M}}$ defines a fiber metric in $u^{-1}TN$. We check if the induced connection $'\nabla$ is compatible with the induced metric u^*h.

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x^i}}h_{\alpha\beta}(u) &=' \nabla_{\frac{\partial}{\partial x^i}}u^*h\left(\frac{\partial}{\partial y^\alpha}\circ u,\frac{\partial}{\partial y^\beta}\circ u\right) && (1.31)\\
&=\nabla'_{\left(du\left(\frac{\partial}{\partial x^i}\right)\right)}h\left(\frac{\partial}{\partial y^\alpha}\circ u,\frac{\partial}{\partial y^\beta}\circ u\right) && (1.32)\\
&=\sum_\gamma\frac{\partial u^\gamma}{\partial x^i}\nabla'_{\frac{\partial}{\partial y^i}}h_{\alpha\beta}\circ u && (1.33)\\
&=0 \quad \text{by compatibility of } \nabla' \text{ with } h && (1.34)
\end{aligned}$$

Now we have connections, ∇^* in TM^* and $'\nabla$ in $u^{-1}TN$, using these two, we introduce a connection on $TM^*\otimes u^{-1}TN$ in a following way.

Let $w \in TM^*$, $W \in u^{-1}TN$ and $X \in TM$. We define,

$$\nabla_X(w \otimes W) = (\nabla^* w) \otimes W + w \otimes ({}'\nabla_X W), \quad X \in \Gamma(TM) \tag{1.35}$$

Linearity is clear. Also,

$$\nabla_X(fw \otimes W) = (\nabla^*_X fw) \otimes W + w \otimes ({}'\nabla_X fW) \tag{1.36}$$

$$= (X(f)w + f\nabla^*_X w) \otimes W + fw \otimes ({}'\nabla_X W) \tag{1.37}$$

$$= X(f)(w \otimes W) + f\nabla_X(w \otimes W). \tag{1.38}$$

This proves that $\nabla : \Gamma(TM^* \otimes u^{-1}TN) \to \Gamma(TM^* \otimes TM^* \otimes u^{-1}TN)$ is a connection in the tensor product $TM^* \otimes u^{-1}TN$. From the definition of ∇ it is clear that this connection is compatible with the fiber metric in $TM^* \otimes u^{-1}TN$.

As discussed previously, the differential of the map $u \in C^\infty(M, N)$ defines a C^∞ section $du \in \Gamma(TM^* \otimes u^{-1}TN)$ in the vector bundle $TM^* \otimes u^{-1}TN$. Consider the covariant differential of du by the connection ∇ in $TM^* \otimes u^{-1}TN$, given by $\nabla du \in \Gamma(TM^* \otimes TM^* \otimes u^{-1}TN)$. This ∇du is called *second fundamental form* of the C^∞ map u. We have

Lemma 1.2.2. *Given $u \in C^\infty(M, N)$ and $X, Y \in \Gamma(TM)$, we have*

$$\nabla du(X, Y) =' \nabla_X du(Y) - du(\nabla_X Y). \tag{1.39}$$

Proof. For $w \in \Gamma\Gamma(TM^*)$, $W \in \Gamma(u^{-1}TN)$

$$(\nabla(w \otimes W))(X,Y) = (\nabla^*_X w \otimes W + w \otimes' \nabla_X W)(Y) \tag{1.40}$$

$$= (Xw(Y) - w(\nabla_X Y)) \otimes W + w(Y) \otimes' \nabla_X W \tag{1.41}$$

$$=' \nabla_X((w \otimes W)(Y)) - (w \otimes W)(\nabla_X Y). \tag{1.42}$$

Hence, we have

$$\nabla du(X,Y) =' \nabla_X du(Y) - du(\nabla_X Y). \tag{1.43}$$

□

This can serve as an alternate definition of ∇du. If we express in coordinates,

$$du = \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \frac{\partial u^\alpha}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^\alpha} \circ u, \tag{1.44}$$

$$\nabla du = \sum_{i,j=1}^{m} \sum_{\alpha=1}^{n} \nabla_i \nabla_j u^\alpha dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^\alpha} \circ u, \tag{1.45}$$

where we have,

Lemma 1.2.3. *For each* $1 \leq i,j \leq m,\ 1 \leq \alpha \leq n$*, the coefficients in* (1.45) *are given by,*

$$\nabla_i \nabla_j u^\alpha = \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \sum_{k=1}^{m} \Gamma^k_{ij} \frac{\partial u^\alpha}{\partial x^k} + \sum_{\beta,\gamma=1}^{n} \Gamma'^\alpha_{\beta\gamma}(u) \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} \tag{1.46}$$

Proof. From the definition of ∇du and (1.45),

$$\nabla_{\frac{\partial}{\partial x^i}} du = \nabla du \left(\frac{\partial}{\partial x^i}, ., . \right) = \sum_{j=1}^{m} \sum_{\alpha=1}^{n} \nabla_i \nabla_j u^\alpha . dx^j \otimes \frac{\partial}{\partial y^\alpha} \circ u \tag{1.47}$$

On the other hand, from the definition of the induced connection ${}'\nabla$ and (1.44),

$$\nabla_{\frac{\partial}{\partial x^i}} du = \nabla_{\frac{\partial}{\partial x^i}} \left(\sum_{j=1}^{m} \sum_{\alpha=1}^{n} \frac{\partial u^\alpha}{\partial x^j} dx^j \otimes \frac{\partial}{\partial y^\alpha} \circ u \right) \tag{1.48}$$

$$= \sum_{j=1}^{m} \sum_{\alpha=1}^{n} \{ \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} dx^j \otimes \frac{\partial}{\partial y^\alpha} \circ u + \frac{\partial u^\alpha}{\partial x^j} \nabla^*_{\frac{\partial}{\partial x^i}} dx^j \tag{1.49}$$

$$\otimes \frac{\partial}{\partial y^\alpha} \circ u + \frac{\partial u^\alpha}{\partial x^j} dx^j \otimes' \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^\alpha} \circ u \} \tag{1.50}$$

$$= \sum_{j=1}^{m} \sum_{\alpha=1}^{n} \left\{ \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \sum_{k=1}^{m} \Gamma_{ij}^k \frac{\partial u^\alpha}{\partial x^k} + \sum_{\beta,\gamma=1}^{n} \Gamma'^{\alpha}_{\beta\gamma}(u) \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} \right\} \tag{1.51}$$

$$.dx^j \otimes \frac{\partial}{\partial y^\alpha} \circ u \text{from (1.26).} \tag{1.52}$$

□

Corollary 1.2.1. *Given* $u \in C^\infty(M, N)$ *and* $X, Y \in \Gamma(TM)$,

$$\nabla du(X, Y) = \nabla du(Y, X) \tag{1.53}$$

Proof. From Lemma 1.2.3, $\nabla_i \nabla_j u^\alpha = \nabla_j \nabla_i u^\alpha$ and then from (1.45). □

Let $E_1, ..., E_m$ be an orthonormal basis for the tangent space $T_x M$ of M at each point $x \in M$. For the second fundamental form ∇du of a C^∞ map u, as readily seen, From orthonormality of basis vectors $E_1, ..., E_m$ and (1.45), trace of this map can be defined as

$$\operatorname{trace} \nabla du(x) = \sum_{i=1}^{m} \nabla du(x)(E_i, E_i) \tag{1.54}$$

$$= \sum_{\alpha=1}^{n} \left(\sum_{i,j=1}^{m} g^{ij} \nabla_i \nabla_j u^\alpha \right) \frac{\partial}{\partial y^\alpha} \circ u \tag{1.55}$$

implying that this trace is independent of the basis functions $E_1, ..., E_m \in T_xM$.

Definition 1.2.1. Given a C^∞ map $u \in C^\infty(M, N)$,

$$\tau(u) = \operatorname{trace} \nabla du \in \Gamma(u^{-1}TN) \tag{1.56}$$

is called the *tension field* of u.

Example 1.2.1. Let $u : \mathbb{R}^m \to \mathbb{R}^n$ be a C^∞ map. Let $(x^1, ..., x^m)$ be a coordinate chart for $\mathbb{R}^m$ and let $(y^1, ..., y^n)$ be a coordinate chart for $\mathbb{R}^n$. Also let $u = (u^1, ..., u^n)$ be the coordinate representation of u. Then

$$\operatorname{trace} \nabla u(x) = \sum_{\alpha=1}^{n} \left(\sum_{i=1}^{m} \frac{\partial^2 u^\alpha}{\partial (x^i)^2} \right) \frac{\partial}{\partial y^\alpha} \circ u(x) \in TN_{u(x)} \tag{1.57}$$

Notice here that the components of the tension field $\tau(u)$ are Laplacians of the coordinate maps.

Due to this example, we can think of the tension field $\tau(u)$ as the 'local Laplacian' of the map.

1.3 The First Variation Formula

With this preparation, we derive a relation between energy of the map $E(u)$ and the tension field $\tau(u)$.

Definition 1.3.1. Consider $u \in C^\infty(M, N)$. A C^∞ map $F : M \times I \to N$ is called a C^∞ *variation* or a *smooth variation* of F provided that

$$F(x, 0) = u(x), \quad x \in M. \tag{1.58}$$

Given a variation F as defined above, we denote

$$u_t(x) = F(x,t), \quad x \in M, t \in I. \tag{1.59}$$

called *variation of* u where $u_0 = u$. When a smooth variation $F = \{u_t\}_{t\in I}$ is given, at each $x \in M$, $u_t(x) = F(x,t) : I \to N$ defines a C^∞ curve in N, passing through $u(x)$ at $t = 0$. Consequently, the set of tangent vectors to these curves at $t = 0$, denoted by

$$V(x) = \frac{d}{dt}\Big|_{t=0} u_t(x) = \frac{\partial F}{\partial t}(x,0) \in T_{u(x)}N, \quad x \in M, \tag{1.60}$$

defines a C^∞ section $V \in \Gamma(u^{-1}TN)$ of the induced bundle $u^{-1}TN$. In other words, $V(x)$ defines a C^∞ vector field in N along the map u. Intuitively, we can think of $V(x)$ as the rate of change of the map u_t at $t = 0$.

Given a smooth variation $F = \{u_t\}_{t\in I}$, we investigate the change of the energy functional E. We have,

$$E(u_t) = \frac{1}{2}\int_M |du_t|^2 \, d\mu_g. \tag{1.61}$$

Theorem 1.3.1 (The first variation formula)**.** *Let $F \in u_{tt\in I}$ be a C^∞ variation of a C^∞ map $u \in C^\infty(M,N)$. Let M be compact. Then*

$$\frac{d}{dt}E(u_t)\Big|_{t=0} = -\int_M \langle V, \tau(u)\rangle d\mu_g, \tag{1.62}$$

where $V = \frac{d}{dt}\Big|_{t=0} u_t$ is a variation vector field of u, and $\tau(u)$ is the tension field of u. $\langle , \rangle$ is the natural fiber metric in the induced bundle $u^{-1}TN$.

Proof. Let $F(x,t) = u_t(x)$ be a map defining a C^∞ variation of u. Consider the vector bundle $T(M \times I)^* \otimes F^{-1}TN$ over the product manifold $M \times I$. As seen above, $T(M \times I)^* \otimes F^{-1}TN$ admits a natural fiber metric $\langle , \rangle$ and a standard connection ∇ compatible with the metric. Under the natural identification $T_{(x,t)}(M \times I) \cong T_x M \oplus T_t I$, we denote the covariant differentiation with respect to the connection ∇ compatible with the metric in the directions $(\partial/\partial x^i, 0) \in T_{(x,t)(M\times I)}$ and $(0, d/dt) \in T_{x,t}(M \times I)$, respectively

$$\nabla_i = \nabla_{(\partial/\partial x^i,0)}, \quad \nabla_t = \nabla_{(0,d/dt)}. \tag{1.63}$$

From the definition of $E(u_t)$,

$$E(u_t) = \frac{1}{2}\int_M \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij} h_{\alpha\beta}(u_t) \left(\frac{\partial u_t^\alpha}{\partial x^i}\right)\left(\frac{\partial u_t^\beta}{\partial x^j}\right) d\mu_g. \tag{1.64}$$

Therefore, since ∇ is compatible with the fiber metric $\langle , \rangle$,

$$\nabla_i g^{jk} h_{\alpha\beta}(u_t) = 0, \qquad \nabla_t g^{jk} h_{\alpha\beta}(u_t) = 0 \tag{1.65}$$

$$\frac{d}{dt}E(u_t)\Big|_{t=0} = \frac{1}{2}\int_M \frac{d}{dt}\left(\sum_{i,j=1}^{m}\sum_{\alpha,\beta=1}^{n} g^{ij} h_{\alpha\beta}(u_t)\frac{\partial u_t^\alpha}{\partial x^i}\frac{\partial u_t^\beta}{\partial x^j}\right)\Big|_{t=0} d\mu_g \tag{1.66}$$

$$= \int_M \sum_{i,j=1}^{m}\sum_{\alpha,\beta=1}^{n} \left(g^{ij} h_{\alpha\beta}(u_t)\nabla_t \frac{\partial u_t^\alpha}{\partial x^i}\frac{\partial u_t^\beta}{\partial x^j}\right)\Big|_{t=0} d\mu_g. \tag{1.67}$$

On the other hand, since $[(0, d/dt), (\partial/\partial x^i, 0)]$, we get

$$\nabla_{(0,d/dt)} du_t \left(\left(\frac{\partial}{\partial x^i}, 0\right)\right) - \nabla_{(\partial/\partial x^i,0)} du_t \left(\left(0, \frac{d}{dt}\right)\right) \tag{1.68}$$

$$= du_t \left(\nabla_{(0,d/dt)}(\partial/\partial x^i, 0) - \nabla_{(\partial/\partial x^i,0)}(d/dt, 0)\right) \quad \text{by 1.2.2 and 1.2.1} \tag{1.69}$$

$$= du_t \left(\left[\left(0, \frac{d}{dt}\right), \left(0, \frac{\partial}{\partial x^i}\right)\right]\right) \tag{1.70}$$

$$= 0 \tag{1.71}$$

This implies that for each $1 \leq i \leq m$, $1 \leq \alpha \leq n$, we get

$$\nabla_t \frac{\partial u_t^\alpha}{\partial x^i} = \nabla_i \frac{\partial u_t^\alpha}{\partial t} \tag{1.72}$$

By writing variational vector field of u in coordinates,ting variational vector field of u in coordinates,

$$V = \sum_{\alpha=1}^{n} V^\alpha \frac{\partial}{\partial y^\alpha} \circ u \quad \text{with } V^\alpha = \frac{\partial u_t^\alpha}{\partial t}\Big|_{t=0}, \tag{1.73}$$

we get,

$$\frac{d}{dt} E(u_t)\Big|_{t=0} = \int_M \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} \left(g^{ij} h_{\alpha\beta}(u_t) \nabla_i \left(\frac{\partial u_t^\alpha}{\partial t}\Big|_{t=0} \right) \frac{\partial u_t^\beta}{\partial x^j} \right) d\mu_g \tag{1.74}$$

$$= \int_M \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} \left(g^{ij} h_{\alpha\beta}(u_t) \nabla_i V \frac{\partial u_t^\beta}{\partial x^j} \right) d\mu_g \tag{1.75}$$

$$= \int_M \langle \nabla V, du \rangle d\mu_g. \tag{1.76}$$

where $\langle , \rangle$ in (1.76) is with respect to the metric $g^{ij} h_{\alpha\beta}$ in $TM^* \otimes u^{-1}TN$. Hence from the next lemma, we obtain the desired first variation formula (1.62). $\square$

Lemma 1.3.2.

$$\int_M \langle \nabla V, du \rangle d\mu_g = - \int_M \langle V, \tau(u) \rangle d\mu_g. \tag{1.77}$$

where on the left hand side, $\langle , \rangle$ is the inner product with respect to the metric $g^{ij} h_{\alpha\beta}$ on $TM^* \otimes u^{-1}TN$, and on the right side, it is with respect to the induced metric $u^* h$ on $u^{-1}TN$.

Proof. Let X be a C^∞ vector field over M given by

$$X = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i} = \sum_{i=1}^{m} \left(\sum_{j=1}^{m} \sum_{\alpha\beta=1}^{n} g^{ij} h_{\alpha\beta}(u) V^\alpha \frac{\partial u^\beta}{\partial x^j} \right) \frac{\partial}{\partial x^i}. \tag{1.78}$$

Denote the covariant derivative of X by

$$\nabla X = \sum_{i,j=1}^{m} \nabla_i X^j \cdot dx^i \otimes \frac{\partial}{\partial x^j}. \tag{1.79}$$

The divergence of X is given by $\operatorname{div} X = \sum_{i=1}^{m} \nabla_i X^i$. Then by compatibility of the connection ∇ with $g^{ij} h_{\alpha\beta}$, (1.44), (1.45) and since

$$\nabla(V \otimes du) = \nabla V \otimes du + V \otimes \nabla du, \tag{1.80}$$

we get,

$$\operatorname{div} X = \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij} h_{\alpha\beta}(u) \nabla_i V^\alpha \frac{\partial u^\beta}{\partial x^j} \tag{1.81}$$

$$+ \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij} h_{\alpha\beta}(u) V^\alpha \nabla_i \nabla_j u^\beta \tag{1.82}$$

$$= \langle \nabla V, du \rangle + \langle V, \tau(u) \rangle. \tag{1.83}$$

Green's theorem

$$\int_M \operatorname{div} X d\mu_g = 0 \tag{1.84}$$

yields the desired result. $\square$

The first variation formula gives the following important relationship between the energy of maps $E(u_t)$ and $\tau(u_t)$.

Corollary 1.3.1. *Given $u \in C^\infty(M, N)$, a necessary and sufficient condition for the first variation of $E(u_t)$ of an arbitrary C^∞ variation $F = \{u_t\}_{t \in I}$ to satisfy*

$$\frac{d}{dt} E(u_t)\Big|_{t=0} = 0 \quad \forall F = \{u_t\}_{t=I}$$

is $\tau(u) \equiv 0$.

Proof. $(\Rightarrow)$ We can take any section $V \in \Gamma(u^{-1}TN)$ in the first variation formula (1.62) can be chosen arbitrarily. $(\Leftarrow)$ Clear from the first variation formula. □

Thus $u \in C^\infty(M, N)$ with $\tau(u) = 0$ is a critical point of the energy functional E.

1.4 Harmonic Maps

We begin with the definition of a harmonic map and give some examples later in the next section. We also derive the coordinate representation of the equation for harmonic maps.

Definition 1.4.1. A C^∞ map $u \in C^\infty(M, N)$ is called a *harmonic map* if its tension field $\tau(u)$ is identically zero; namely,

$$\tau(u) = \operatorname{trace} \nabla du \equiv 0 \tag{1.85}$$

holds in M. (1.85) is called the *equation for harmonic maps*.

When M is compact, a harmonic map u is also a critical point of the energy functional E. In fact, from Corollary 1.3.1, the map $u \in C^\infty(M, N)$ being harmonic means that

$$\frac{d}{dt} E(u_t)\Big|_{t=0} = 0 \tag{1.86}$$

holds for arbitrary C^∞ variation $F = \{u_t\}_{t \in I}$ of u.

We now derive a coordinate representation of the equation for harmonic maps. Let (x^i) and (y^α) denote local coordinate systems in M and N, respectively. With these local coordinates, we express the map u by

$$u(x) = (u^1(x^1, ..., x^m), ..., u^n(x^1, ..., x^m)) = (u^\alpha(x^i)), \tag{1.87}$$

and denote the tension field $\tau(u)$ of u by

$$\tau(u) = \sum_{\alpha=1}^{n} \tau(u)^\alpha \frac{\partial}{\partial y^\alpha} \circ u \in \Gamma(u^{-1}TN). \tag{1.88}$$

where

$$\tau(u)^\alpha = \sum_{i,j=1}^{m} g^{ij} \left\{ \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \sum_{k=1}^{m} \Gamma_{ij}^k \frac{\partial u^\alpha}{\partial x^k} + \sum_{\beta,\gamma=1}^{n} \Gamma'^\alpha_{\beta\gamma}(u) \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} \right\} \tag{1.89}$$

$$= \Delta u^\alpha + \sum_{i,j=1}^{m} \sum_{\beta,\gamma}^{n} g^{ij} \Gamma'^\alpha_{\beta\gamma}(u) \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j}. \tag{1.90}$$

Here Γ^i_{jk} and $\Gamma'^\alpha_{\beta\gamma}$ respectively represent the connection coefficients of the Levi-Civita connection in M and N, and Δ is the Laplacian in M. Therefore, from (1.85) and (1.90), we get the coordinate equation of the harmonic maps

$$\Delta u^\alpha + \sum_{i,j=1}^{m} \sum_{\beta\gamma}^{n} g^{ij} \Gamma'^\alpha_{\beta\gamma}(u) \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} = 0, \quad 1 \le \alpha \le n. \tag{1.91}$$

Chapter 2

The Heat Flow Method

In this chapter we state and prove existence theorem for harmonic maps between two manifolds. Furthermore we prove that any continuous map can be free homotopically deformed to a harmonic map provided that certain conditions are satisfied.

2.1 The Eells Sampson Theorem

The goal of this section is to present the statement of Eell and Sampson's theorem, which is fundamental in the theory of harmonic mappings between Riemannian manifolds. We discuss the statement and various definitions associated with it, and discuss their implications. The proof is given in later sections.

Eells Sampson's Theorem 1. *Let (M, g) and (N, h) be compact Riemannian manifolds. Assume that (N,h) is of non-positive curvature. Then for any $f \in C^\infty(M, N)$, there is a harmonic map $u_\infty : M \to N$ free-homotopic to f.*

Unlike the existence theorem of closed geodesics, the condition on sectional curvature is necessary. For example, Eells and Woods [EW76] show that any map $f : T^2 \to S^2$ of mapping degree ± 1 from the 2D torus T^2 to 2D sphere S^2 is not free homotopic to a harmonic map, regardless of the Riemannian metric g, h on S^2 and T^2.

For the existence proof of the harmonic maps, the direct variational technique encounters some difficulties because of the nonlinearity of the system of equations, contrary to the fact that the defining equations for geodesics is a system of linear differential equations. However Eells and Sampson were successful in proving Theorem 1 by the *heat flow method*.

2.2 The Heat Flow Method

In this section we discuss the heat flow method which not only proves the existence of a harmonic map but also tells us how to find one. The idea is to smoothly deform a given initial map $f \in C^\infty(M, N)$ in a 'best' possible way to minimize the energy functional $E(u_t)$. Let $u \in C^\infty(M, N)$ and let $F = \{u_t\}_{t\in I},\ I = (-\epsilon, \epsilon)$ be its first variation. Then the variation vector field $V = \frac{d}{dt}u_t|_{t=0}$. Let $\tau(u)$ be the tension field of u, $\langle , \rangle$ be the natural fiber metric in the induced vector bundle $u^{-1}TN$.

The rate of change of the energy functional E is given by the first variation formula for the energy functional $E(u_t)$:

$$\frac{d}{dt}E(u_t)|_{t=0} = -\int_M \langle V, \tau(u)\rangle d\mu_g. \tag{2.1}$$

If we consider $S = C^\infty(M, N)$ as a (infinite dimensional) manifold, u_t can be regarded as a curve on S and the variation vector field $\frac{d}{dt}u_t$ can be regarded as a vector in tangent space T_uS. In this space, we define the inner product $\langle\langle W_1, W_2\rangle\rangle = \int_M \langle W_1, W_2\rangle d\mu_g$. Also the derivative of the function $E(u_t)$ on S in the direction of $V = \frac{d}{dt}u_t|_{t=0}$ is given by $dE_u(V) = \frac{d}{dt}E(u_t)|_{t=0}$. Therefore, the first variation formula can be written as

$$dE_u(V) = -\langle\langle \tau(u), V\rangle\rangle. \tag{2.2}$$

Since this is true for any variation vector field $V \in T_uS$, by definition of gradient on manifold, we have

$$\tau(u) = -(\operatorname{grad} E)(u). \tag{2.3}$$

Consequently, a harmonic map u which is a critical point of the energy functional is a zero of the gradient vector field $\operatorname{grad} E$. We may say that the functional E decreases in the direction of $-\operatorname{grad} E$. The *heat flow method* tries to deform a map $u_0 = f$ and find a *flow* u_t such that

$$\frac{\partial u_t}{\partial t} = \tau(u_t) \tag{2.4}$$

i.e. it tries to move $u_t \in S$ in the direction of the negative gradient of E and hopes that critical point of E is attained. So a sufficient condition for the existence of the harmonic map free homotopic to $u_0 = f$ is that $\tau(u_\infty) = \operatorname{grad} E(u_\infty) = 0$. With this in mind, we study the following initial value problem. For a given $f \in C^\infty(M, N)$ and $T > 0$ and assuming that u is continuous, we would like to know if the following system of equations has a solution.

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \tau(u(x,t)), \quad (x,t) \in M \times (0,\infty), \\ u(x,0) = f(x). \end{cases} \tag{2.5}$$

This is a non-linear parabolic system of equations. To prove Theorem (1), we ask:

1. For any initial value $f \in C^\infty(M, N)$, does (2.5) has a solution?

2. If yes, then is u_∞ harmonic and free homotopic to $u_0 = f$?

First lets assume that 1 holds and lets try to answer 2. The key to analyzing 2 is to study *how fast* does $E(u_t)$ change if u_t satisfies (2.5). Lets first define a few quantities which measure this rate of change.

$$
\begin{aligned}
e(u_t) &= \frac{1}{2}|du_t|^2 && \text{Harmonic energy density,} && (2.6)\\
E(u_t) &= \int_M e(u_t)d\mu_g && \text{Harmonic energy,} && (2.7)\\
\kappa(u_t) &= \frac{1}{2}|\frac{\partial u_t}{\partial t}| && \text{Kinetic energy density} && (2.8)\\
K(u_t) &= \int_M \kappa(u_t)d\mu_g && \text{Kinetic energy} && (2.9)
\end{aligned}
$$

To study the rate of change of $E(u_t)$, we study these quantities. In particular we want to know what is $\frac{\partial e(u_t)}{\partial t}$ and what is $\frac{\partial \kappa(u_t)}{\partial t}$ and how are they related to the geometry of M and N. The Weitzenböck formula gives us an important relation between these and curvatures of the manifolds M and N.

For the proof of Weizenböck formulas, we need a notion of second order covariant derivatives. The connection

$$\nabla : \Gamma(TM^* \otimes u^{-1}TN) \to \Gamma(TM^* \otimes TM^* \otimes u^{-1}TN) \tag{2.10}$$

is compatible with the fiber metric $\langle , \rangle$ in $TM^* \otimes u^{-1}TN$.

Lemma 2.2.1. *Given $T \in \Gamma(TM^* \otimes u^{-1}TN)$ and $X, Y, Z \in \Gamma(TM)$, we have*

$$(\nabla\nabla T)(X, Y, Z) = (\nabla_X \nabla T)(Y, T) = (\nabla_X(\nabla_Y T))(Z) - (\nabla_{\nabla_X Y} T)(Z) \tag{2.11}$$

Proof.

$$(\nabla\nabla T)(X,Y,Z) = (\nabla_X \nabla T)(Y,T) \tag{2.12}$$

$$= \nabla_X(\nabla T(Y,Z)) - \nabla T(\nabla_X Y, Z) - \nabla T(Y, \nabla_X Z) \tag{2.13}$$

$$= \nabla_X((\nabla_Y T)(Z)) - (\nabla_{\nabla_X Y} T)(Z) - (\nabla_Y T)(\nabla_X Z) \tag{2.14}$$

$$= (\nabla_X(\nabla_Y T))(Z) - (\nabla_{\nabla_X Y} T)(Z). \tag{2.15}$$

$\square$

We need an important identity called the *Ricci identity*. First we prove a lemma which we use for proving the identity.

Lemma 2.2.2. *Let $T \in \Gamma(TM^* \otimes u^{-1}TN)$ and $X, Y, Z \in \Gamma(TM)$. Denote ∇ the connection in $TM^* \otimes u^{-1}TN$ and by $'\nabla$ the induced connection in $u^{-1}TN$. Also denote by R^M, R^N the curvature tensors of M, N respectively. Set*

$$R^{\nabla}(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \tag{2.16}$$

$$R^{'\nabla}(X,Y) =' \nabla'_X \nabla_Y -' \nabla'_Y \nabla_X -' \nabla_{[X,Y]}. \tag{2.17}$$

Then,

$$(R^{\nabla}(X,Y)T)(Z) = R^{'\nabla}(X,Y)(T(Z)) - T(R^M(X,Y)Z). \tag{2.18}$$

Proof.

$$'\nabla_Y(T(Z)) = (\nabla_Y)(Z) + T(\nabla_Y Z), \tag{2.19}$$

$$'\nabla'_X \nabla_Y(T(Z)) = (\nabla_X \nabla_Y T)(Z) + (\nabla_Y T)(\nabla_X Z) \tag{2.20}$$

$$+ (\nabla_X T)(\nabla_Y Z) + (\nabla_X T)(\nabla_Y Z). \tag{2.21}$$

Similarly computing $-{}'\nabla'_Y\nabla_X(T(Z))$ and $-{}'\nabla_{[X,Y]}(T(Z))$ and adding them together, we get

$$R^{'\nabla}(X,Y)(T(Z)) = (R^{\nabla}(X,Y)T)(Z) + T(R^M(X,Y)Z). \tag{2.22}$$

□

Proposition 2.2.3 (Ricci identity)**.** *Let* $T \in \Gamma(TM^* \otimes u^{-1}TN)$*. With respect to the local coordinates* (x^i)*,* (y^α) *on* M *and* N*, express*

$$T = \sum_{i=1}^{m}\sum_{\alpha=1}^{n} T_i^\alpha dx^i \otimes \frac{\partial}{\partial y^\alpha}\circ u, \tag{2.23}$$

$$\nabla\nabla T = \sum_{i,j,k=1}^{m}\sum_{\alpha=1}^{n} \nabla_i\nabla_j T_k^\alpha dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^\alpha}\circ u. \tag{2.24}$$

Then,

$$\nabla_i\nabla_j T_k^\alpha - \nabla_j\nabla_i T_k^\alpha = -\sum_{l=1}^{m} R_{ijk}^{M^l} T_l^\alpha + \sum_{\beta,\gamma,\delta=1}^{\alpha} R_{\beta,\gamma,\delta}^{N^\alpha}\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j} T_k^\delta. \tag{2.25}$$

Proof. Using the notation in Lemma 2.2.2 and the definition of the induced connection ${}'\nabla$ in vector bundle $u^{-1}TN$, from Lemma 2.2.2, we get

$$\begin{aligned}
&\left(R^{\nabla}\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)T\right)\left(\frac{\partial}{\partial x^k}\right)\\
&= R^{'\nabla}\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\left(T\left(\frac{\partial}{\partial x^k}\right)\right) - T\left(R^M\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}\right)\\
&= \sum_\alpha\left(\sum_{\beta,\gamma,\delta} R_{\beta,\gamma,\delta}^{N^\alpha}\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j}T_k^\delta - \sum_l R_{ijk}^{M^l}T_l^\alpha\right)\frac{\partial}{\partial y^\alpha}\circ u
\end{aligned} \tag{2.26}$$

On the other hand, since we have

$$\nabla\nabla T\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k}\right) = \sum_\alpha \nabla_i\nabla_j T_k^\alpha\frac{\partial}{\partial y^\alpha}\circ u, \tag{2.27}$$

The conclusion follows from the definition of R^{∇} given in (2.16), from fact that partial derivatives commute, and then comparing coefficients. $\square$

Consider the second order covariant differential as given by Lemma 2.2.1. The connection on the tensor ∇du is given by

$$\nabla\nabla du(X,Y,Z) = (\nabla_X(\nabla_Y du))(Z) - (\nabla_{\nabla_X Y} du)(Z). \tag{2.28}$$

In local coordinate systems (x^i) and (y^i) of M and N, respectively, we express du, ∇du and $\nabla\nabla du$ and the curvature tensors R^M and R^N by

$$du = \sum_{i=1}^{m}\sum_{\alpha=1}^{n} \frac{\partial u^\alpha}{\partial x^i} \cdot dx^i \otimes \frac{\partial}{\partial y^\alpha} \circ u, \tag{2.29}$$

$$\nabla du = \sum_{i,j=1}^{m}\sum_{\alpha=1}^{n} \nabla_i \nabla_j u^\alpha \cdot dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^\alpha} \circ u, \tag{2.30}$$

$$\nabla\nabla du = \sum_{i,j,k=1}^{m}\sum_{\alpha=1}^{n} \nabla_i \nabla_j \nabla_k u^\alpha \cdot dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^\alpha} \circ u, \tag{2.31}$$

$$R^M = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = \sum_{l=1}^{m} R^{M^l}_{ijk} \frac{\partial}{\partial x^l}, \tag{2.32}$$

$$R^N = \left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right)\frac{\partial}{\partial y^\gamma} = \sum_{\delta=1}^{n} R^{M^\delta}_{\alpha\beta\gamma} \frac{\partial}{\partial y^\delta} \tag{2.33}$$

Corollary 2.2.1. *In* (2.31),

$$\nabla_i\nabla_j\nabla_k u^\alpha - \nabla_j\nabla_i\nabla_k u^\alpha = -\sum_{l=1}^{m} R^{M^l}_{ijk}\frac{\partial u^\alpha}{\partial x^l} + \sum_{\beta\gamma\delta=1}^{n} R^{N^\alpha}_{\beta\gamma\delta}\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j}\frac{\partial u^\delta}{\partial x^k}. \tag{2.34}$$

Proof. This is a special case of the Ricci identity in Proposition (2.2.3) with $T = du$. $\square$

Proposition 2.2.4 (Weizenböck formulas)**.** *Let* $u \in C^0(M \times [0,T), N) \cap C^\infty(M \times (0,T), N)$ *be a solution to* (2.5)*. Then we have in* $M \times (0,T)$*,*

$$\begin{aligned}\frac{\partial e(u_t)}{\partial t} &= \Delta e(u_t) - |\nabla\nabla u_t|^2 - \sum_{i=1}^{m}\left\langle du_t(\sum_{j=1}^{m} Ric^M(e_i,e_j)e_j), du_t(e_i)\right\rangle \\ &\quad + \sum_{i,j=1}^{m}\left\langle R^N(du_t(e_i), du_t(e_j))du_t(e_j), du_t(e_i)\right\rangle\end{aligned} \tag{2.35}$$

and,

$$\frac{\partial \kappa(u_t)}{\partial t} = \Delta\kappa(u_t) - |\nabla\frac{\partial u_t}{\partial t}| + \sum_{i=1}^{m}\left\langle R^N(du_t(e_i), \frac{\partial u_t}{\partial t})\frac{\partial u_t}{\partial t}, du_t(e_i)\right\rangle. \tag{2.36}$$

Proof. Consider the induced connection ∇ compatible with the natural fiber metric $\langle , \rangle$ in the vector bundle $(TM \times (0,T))^* \otimes u^{-1}TN$ over $M \times (0,T)$. Using this connection, we denote the covariant differentiations in the directions $(\partial/\partial x^i, 0) \in T_{(x,t)}(M \times (0,T))$ and $(0, d/dt) \in T_{(x,t)}(M \times (0,T))$, respectively, by

$$\nabla_i = \nabla_{(\partial/\partial x^i, 0)}, \qquad \nabla_t = \nabla_{(0,d/dt)}. \tag{2.37}$$

Since ∇ is compatible with the fiber metric $\langle , \rangle$, we see that

$$\nabla_i g^{jk} h_{\alpha\beta}(u_t) = 0, \qquad \nabla_t g^{jk} h_{\alpha\beta}(u_t) = 0. \tag{2.38}$$

Therefore,

$$\frac{\partial e(u_t)}{\partial t} = \nabla_t \left(\frac{1}{2} \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij} h_{\alpha\beta}(u_t) \frac{\partial u_t^\alpha}{\partial x^i} \frac{\partial u_t^\beta}{\partial x^j} \right) \tag{2.39}$$

$$= \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij} h_{\alpha\beta}(u_t) (\nabla_t \frac{\partial u_t^\alpha}{\partial x^i}) \frac{\partial u_t^\beta}{\partial x^j}. \tag{2.40}$$

From (1.72), we get

$$\nabla_t \frac{\partial u_t^\alpha}{\partial x^i} = \nabla_i \frac{\partial u_t^\alpha}{\partial t}, \quad 1 \leq i \leq m,\, 1 \leq \alpha \leq n. \tag{2.41}$$

Consequently, we get

$$\frac{\partial e(u_t)}{\partial t} = \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij} h_{\alpha\beta}(u_t) (\nabla_i \frac{\partial u_t^\alpha}{\partial x^i}) \frac{\partial u_t^\beta}{\partial x^j} \tag{2.42}$$

$$= \sum_{i=1}^{m} \left\langle \nabla_{e_i} \frac{\partial u_t}{\partial t}, du_t(e_i) \right\rangle. \tag{2.43}$$

Since we have

$$\nabla_k \nabla_l e(u_t) = \nabla_k \nabla_l \left(\frac{1}{2} \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij} h_{\alpha\beta}(u_t) \frac{\partial u_t^\alpha}{\partial x^i} \frac{\partial u_t^\beta}{\partial x^j} \right) \tag{2.44}$$

$$= \nabla_k \left(\sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij} h_{\alpha\beta}(u_t) \nabla_l \nabla_i u_t^\alpha \frac{\partial u_t^\beta}{\partial x^j} \right) \tag{2.45}$$

$$= \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij} h_{\alpha\beta}(u_t) \left(\nabla_k \nabla_l \nabla_i u_t^\alpha \frac{\partial u_t^\beta}{\partial x^j} + \nabla_l \nabla_i u_t^\alpha \nabla_k \nabla_j u_t^\beta \right), \tag{2.46}$$

noting $\nabla_l \nabla_i u_t^\alpha = \nabla_i \nabla_l u_t^\alpha, 1 \leq i, l \leq m, 1 \leq \alpha \leq m$ as given in Corollary 1.2.1, we get

$$\Delta e(u_t) = \sum_{k,l=1}^{m} g^{kl} \nabla_k \nabla_l e(u_t) \tag{2.47}$$

$$= \sum_{i,j,k,l=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij} g^{kl} h_{\alpha\beta}(u_t) \nabla_k \nabla_i \nabla_l u_t^\alpha \frac{\partial u_t^\beta}{\partial x^j} + |\nabla \nabla u_t|^2 . \tag{2.48}$$

Now applying Corollary 2.2.1, we get

$$\Delta e(u_t) = \sum_{i,j=1}^{m} \sum_{\alpha\beta=1}^{n} g^{ij} h_{\alpha\beta}(u_t) \nabla_i \left(\sum_{j,k=1}^{m} g^{kl} \nabla_k \nabla_l u_t^\alpha \right) \frac{\partial u_t^\beta}{\partial x^j} + |\nabla \nabla u_t|^2 \tag{2.49}$$

$$- \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij} h_{\alpha\beta}(u_t) \left\{ \sum_{r=1}^{m} \left(\sum_{k,l=1}^{m} g^{kl} R_{kil}^{M^r} \right) \frac{\partial u_t^\alpha}{\partial x^r} \right\} \frac{\partial u_t^\beta}{\partial x^j} \tag{2.50}$$

$$+ \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta}(u_t) \left(\sum_{i,j,k,l=1}^{m} \sum_{\gamma\delta\epsilon=1}^{n} g^{ij} g^{kl} R_{\gamma\delta\epsilon}^{N^\alpha} \frac{\partial u_t^\gamma}{\partial x^k} \frac{\partial u_t^\delta}{\partial x^i} \frac{\partial u_t^\epsilon}{\partial x^l} \frac{\partial u_t^\beta}{\partial x^j} \right) \tag{2.51}$$

$$= \sum_{i=1}^{m} \langle \nabla_{e_i} \tau(u_t), du_t(e_i) \rangle + |\nabla \nabla u_t|^2 \tag{2.52}$$

$$+ \sum_{i=1}^{m} \left\langle du_t \left(\sum_{j=1}^{m} Ric^M(e_i, e_j) e_j \right), du_t(e_i) \right\rangle \tag{2.53}$$

$$- \sum_{i,j=1}^{m} \left\langle R^N(du_t(e_i), du_t(e_j)) du_t(e_j), du_t(e_i) \right\rangle . \tag{2.54}$$

We know that $\partial u_t / \partial t = \tau(u_t)$ holds since by hypothesis, u_t is a solution to (2.5). Substituting this in the above equation for $\Delta e(u_t)$ and then comparing with (2.43) yields the desired identity. $\square$

We use The Weizenböck formulas to estimate the rates of change of the harmonic energy $E(u_t)$.

Corollary 2.2.2. *Let $u : M \times [0,T) \to N$ be a solution to the parabolic system of equations* (2.5)*, then*

1. *If N is of non-positive curvature $K_N \leq 0$, and if, furthermore, there exists a constant C such that $Ric^M \geq -Cg$, then*

$$\frac{\partial e(u_t)}{\partial t} \leq \Delta e(u_t) + 2Ce(u_t). \tag{2.55}$$

2. *If N is of non positive curvature $K_N \leq 0$, then*

$$\frac{\partial \kappa(u_t)}{\partial t} \leq \Delta \kappa(u_t). \tag{2.56}$$

Proof. (1) Note that the fourth term of the right hand side of (2.35) is negative because K_N is negative. Also, since $Ric^M \geq -Cg$, we have

$$du_t(\sum_{j=1}^{m} Ric^M(e_i, e_j)e_j) \geq -C du_t(e_i). \tag{2.57}$$

Therefore the third term in (2.35)

$$-\sum_{i=1}^{m} \left\langle du_t(\sum_{j=1}^{m} Ric^M(e_i, e_j)e_j), du_t(e_i) \right\rangle \geq C \left\langle du_t(e_i), du_t(e_i) \right\rangle \tag{2.58}$$

$$= 2e(u_t). \tag{2.59}$$

Here the inner product was in $\mathrm{Hom}(M, N) = \Gamma(TM^*) \otimes \Gamma(u^-1TN)$ The second term in (2.35) is negative which leads to the desired inequality (2.55).

(2) follows from (2.36). □

The above corollary is remarkable since it puts bounds on rates of change of harmonic and kinetic energy densities $e(u_t)$ and $\kappa(u_t)$. Also note that, if M is compact, there always exists $C \in \mathbb{R}$ as required in 1. From these bounds on rates of change of densities, we get bounds on the rate of change of harmonic energy functional itself.

Proposition 2.2.5. *Let $u : M \times [0,T) \to N$ be a solution to the parabolic system of equations* (2.5)*. Then the following holds,*

1. *$E(u_t)$ is a monotone non-increasing function, i.e.*

$$\frac{d}{dt}E(u_t) = -2K(u_t) \leq 0. \tag{2.60}$$

2. *If N is of non-positive curvature $K_N \leq 0$, then*

$$\frac{d^2}{dt^2}E(u_t) = -2\frac{d}{dt}K(u_t) \geq 0. \tag{2.61}$$

Proof. (2.60) The first variational formula gives

$$\frac{d}{dt}E(u_t) = -\int_M \langle \frac{\partial u_t}{\partial t}, \tau(u_t)\rangle d\mu_g \tag{2.62}$$

$$= -\int_M \langle \frac{\partial u_t}{\partial t}, \frac{\partial u_t}{\partial t}\rangle d\mu_g = -|\frac{\partial u_t}{\partial t}| \tag{2.63}$$

$$= -2K(u_t). \tag{2.64}$$

(2.61) Differentiating one more time,

$$\frac{d}{dt}K(u_t) = \frac{d}{dt}\int_M \kappa(u_t) d_m u_g = \int_M \frac{\partial \kappa(u_t)}{\partial t} \leq \int_M \Delta\kappa(u_t) d\mu_g = 0. \tag{2.65}$$

by 2.56 and Green's theorem. □

This proves that if N has non-positive curvature, then the harmonic energy $E(u_t)$ is a convex function and the kinetic energy $K(u_t)$ is a monotone non-increasing function.

Since $E(u_t) \geq 0$, and it is monotonically decreasing, the rate of change $\frac{d}{dt}E(u_t) = K(u_t) \to 0$ must hold. Therefore from the definition of $K(u_t)$, $\partial u_t/\partial t \to 0$ as $t \to 0$. Since we assumed u_t is a solution of (2.5), $\tau(u_t) \to 0$. In other words, u_t converges to a harmonic map u_∞. Thus we have proved that if 2.5 has a solution, then the harmonic map exists. Now, we prove existence of solution to (2.5).

Chapter 3

Existence of Local and Global Solutions

In this chapter we prove existence of a local solution to the heat flow (2.5). We assume that M and N are compact. We use many results from the theory of PDEs. The results that are used often are summarized in the Appendix A.1.

3.1 Existence of Local Solutions

First we note that

Theorem 3.1.1. *If a C^2 differentiable map $u : M \to N$ satisfies the equation for harmonic maps*

$$\tau(u) = 0 \tag{3.1}$$

then, u is a C^∞ map.

Proof. Without loss of generality, we may verify this at each point $x \in M$. Let V be a coordinate neighborhood about $x \in M$ and let W be a coordinate neighborhood about $u(x)$ such that $u(V) \subset W$. In the local coordinates (x^i) in V and (y^α) in W, the equation for harmonic maps (1.90) is expressed as

$$\Delta u^\alpha = -\sum_{i,j=1}^{m} \sum_{\beta,\gamma=1}^{n} g^{ij} \Gamma'^{\alpha}_{\beta,\gamma} \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j}. \tag{3.2}$$

where Δ is the Laplace operator of M and $\Gamma'^{\alpha}_{\beta,\gamma}$ are the connection coefficients of the Levi-Civita connection on N. Suppose u is C^2. Therefore the right hand side is a C^1 function. In particular it is σ-Hölder continuous for $0 < \sigma < 1$. Consequently, u is of $C^{2+\sigma}$ from Theorem A.2.2 on differentiability for the solutions to linear elliptic partial differential equations. Therefore, the right hand side must be $C^{1+\sigma}$. Hence u must be $C^{2+2\sigma}$ from the same theorem. Repeating the argument, we see that u is C^∞. $\square$

This tells us that we only have to show the existence of a C^2 solution.

In this section, we seek a *local time-dependent solution* to the following initial value problem in (3.3). Namely, for sufficiently small T, the following has a solution.

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \tau(u(x,t)), \quad (x,t) \in M \times (0,T), \\ u(x,0) = f(x) \end{cases} \tag{3.3}$$

To do this, we first simplify the problem to the Euclidean case by Nash's embedding theorem which shows that an arbitrary compact Riemannian manifold can be isometrically embedded in Euclidean space of sufficiently high dimension. Therefore, we may assume, without loss of generality, that (N,h) is realized as a submanifold of the q-dimensional Euclidean space $\mathbb{R}^q$ for a large enough integer q, and that the Riemannian metric h is the induced metric from $\mathbb{R}^q$. Let

$$i : N \to \mathbb{R}^q \tag{3.4}$$

denote such an isometric embedding, and let $\tilde{N}$ be a tubular neighborhood of the submanifold $i(N) \subset \mathbb{R}^q$ in $\mathbb{R}^q$, i.e. for sufficiently small $\epsilon > 0$, $\tilde{N}$ is an open subset

$$\tilde{N} = \left\{ (x,v) |\, x \in i(N),\, v \in T_x i(N)^{\perp},\, |v| < \epsilon \right\}. \tag{3.5}$$

In the tubular neighborhood $\tilde{N}$, let $\pi : \tilde{N} \to i(N)$ denote the projection map which assigns to each $z \in \tilde{N}$ the closest point in $i(N)$ from z.

Now the idea is to transform the initial value problem (3.3) to an equivalent initial value problem in the tubular neighborhood and prove the existence of solution. Since the tubular neighborhood is a submanifold of the Euclidean space $\mathbb{R}^q$, this simplifies things to some extent.

Let $u : M \times [0, T) \to \tilde{N}$ be a map from $M \times [0, t)$ into $\tilde{N} \subset \mathbb{R}^q$. Therefore u can be regarded as a $\mathbb{R}^q$ valued function. Consider the initial value problem

$$\begin{cases} (\Delta - \frac{\partial}{\partial t})u(x,t) = \Pi(u)(du, du)(x,t), \quad (x,t) \in M \times (0,T), \\ u(x,0) = i \circ f(x). \end{cases} \tag{3.6}$$

Here Δ is the Laplace operator in M, and $\Pi(u)(du, du)$ is a vector in $\mathbb{R}^q$ defined as follows. Let $(z^A)_{1 \le A \le q}$ be the standard coordinate system of $\mathbb{R}^q$, and let $(x^i)_{1 \le i \le m}$ be the local coordinate system of M. Then we express the projection map $\pi : \tilde{N} \to i(N)$ and $u : M \times [0, t) \to \tilde{N}$ as

$$\pi(z) = (\pi^1(z^1, ..., z^q), ..., \pi^q(z^1, ..., z^q)) = (\pi^A(z^B)), \tag{3.7}$$

$$u(x,t) = (u^1(x^1, ..., x^m, t), ..., u^q(x^1, ..., x^m, t)) = (u^A(x^i, t)). \tag{3.8}$$

Then we define the components of the vector $\Pi(u)(du, du)$ by

$$\sum_{i,j}^{m} \sum_{B,C=1}^{q} g^{ij} \frac{\partial^2 \pi^A}{\partial z^B \partial z^C}(u) \frac{\partial u^B}{\partial x^i} \frac{\partial u^C}{\partial x^j}, \quad 1 \le A \le q. \tag{3.9}$$

Similar to (1.89), and noting that π is in Euclidean space and hence the connection coefficients are zero, we can interpret Π as

$$\Pi(u)(du, du) = \operatorname{trace} \nabla d\pi(du, du). \tag{3.10}$$

The following proposition proves that instead of finding a solution to the initial value problem (3.3), we can equivalently seek a solution to (3.6).

Proposition 3.1.2. *Let $u \in C^0(M \times [0,T), \tilde{N}) \cap C^{2,1}(M \times (0,T), \tilde{N})$. If u is a solution to the initial value problem* (3.6)*, then $u(M \times [0,T)) \subset i(N)$ holds, and u is a solution to the initial value problem* (3.3)*. The converse also holds true.*

Proof. Suppose $u \in C^0(M \times [0,T), \tilde{N}) \cap C^{2,1}(M \times (0,T), \tilde{N})$ is a solution of (3.6). First we verify that $u(M \times [0,t)) \subset i(N)$ holds. Define a map $\rho : \tilde{N} \to \mathbb{R}^q$ by

$$\rho(z) = z - \pi(z), \quad z \in (\tilde{N}) \tag{3.11}$$

and define $\phi : M \times [0,T) \to \mathbb{R}$ by

$$\phi(x,t) = |\rho(u(x,t))|^2, \quad (x,t) \in M \times [0,T). \tag{3.12}$$

From the definition of π, $\rho(z) = 0$ is equivalent to $z \in i(N)$. Hence, it is sufficient to show that $\phi(x,t) = 0$ since it implies that $u(M \times [0,T) \in i(N))$. Since $u(x,0) = i \circ f(x) \in i(N)$, we see that $\phi(x,0) = 0$. Since u is a solution of (3.6), we get

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial t}\langle \rho(u), \rho(u)\rangle = 2\left\langle d\rho(\frac{\partial u}{\partial t}), \rho(u)\right\rangle \tag{3.13}$$

$$= 2\langle d\rho(\Delta u - \Pi(u)(du,du)), \rangle, \tag{3.14}$$

$$\Delta\phi = \Delta\langle \rho(u), \rho(u)\rangle \tag{3.15}$$

$$= 2\langle \Delta\rho(u), \rho(u)\rangle + 2|\nabla\rho(u)|^2, \tag{3.16}$$

where the inner product $\langle , \rangle$ and ∇ are in $\mathbb{R}^q$ as usual. Δ is usual Laplacian in $\mathbb{R}^q$. We know that the following identity holds. [1]

$$\Delta\rho(u) = d\rho(\Delta u) + \operatorname{trace}\nabla d\rho(du,du). \tag{3.20}$$

---

[1]Let M_1, M_2, M_3 be Riemannian manifolds, and let $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_3$ be smooth maps. From Lemma 1.2.2 and definition of induced connection, for $X, Y \in \Gamma(TM_1)$

$$\nabla d(f_2 \circ f_1)(X,Y) = \nabla_X(df_2 \circ df_1(Y)) - (df_2 \circ df_1)(\nabla_X Y) \tag{3.17}$$

$$= (\nabla_{df_1(X)} df_2)(df_1(Y)) + df_2(\nabla_X(df_1(Y))) - df_2 \circ df_1(\nabla_X Y) \tag{3.18}$$

$$= \nabla df_2(df_1(X), df_1(Y)) + df_2(\nabla df_1(X,Y)). \tag{3.19}$$

Taking trace, we get

$$\tau(f_2 \circ f_1) = \operatorname{trace}\nabla df_2(df_1, df_1) + df_2(\tau(f_1))$$

. When M_3 is Euclidean, then recalling that tension field is the Laplacian in that case, we get

$$\Delta(f_2 \circ f_1) = \operatorname{trace}\nabla df_2(df_1, df_1) + df_2(\Delta(f_1))$$

.

Since $\pi(z) + \rho(z) = z$ from the definition, we have $d\pi + d\rho = id$ the identity map, and therefore, $\nabla d\pi + \nabla d\rho = 0$. Also note that images of $d\pi$ and ρ are orthogonal. Therefore, we get

$$\Delta\phi = 2\langle d\rho(\Delta u) - \operatorname{trace}\nabla d\pi(du, du), \rho(u)\rangle + 2|\nabla\rho(u)|^2 \tag{3.21}$$

$$= 2\langle d\rho(\Delta u) - \Pi(du, du), \rho(u)\rangle + 2|\nabla\rho(u)|^2 \quad \text{from (3.10)} \tag{3.22}$$

$$= 2\langle d\rho(\Delta u) - (d\rho + d\pi)\Pi(du, du), \rho(u)\rangle + 2|\nabla\rho(u)|^2 \tag{3.23}$$

$$= 2\langle d\rho(\Delta u - \Pi(du, du)), \rho(u)\rangle + 2|\nabla\rho(u)|^2. \tag{3.24}$$

From (3.14), we get

$$\frac{\partial\phi}{\partial t} = \Delta\phi - 2|\nabla\rho(u)|^2. \tag{3.25}$$

Green's theorem yields, for each $t \in (0, T)$,

$$\frac{d}{dt}\int_M \phi(\cdot, t)d\mu_g = \int_M \frac{\partial\phi}{\partial t}(\cdot, t)d\mu_g = -2\int_M |\nabla\rho(u)|^2 d\mu_g \leq 0. \tag{3.26}$$

Therefore,

$$0 \leq \int_M \phi(\cdot, t) \leq \int_M \phi(\cdot, 0) = 0 \quad \text{since } \phi(\cdot, t) = 0 \tag{3.27}$$

implying that $\phi(x, t) \equiv 0$. Therefore $u(M \times [0, T)) \subset i(N)$ as required.

Next we verify that u is also solution of the first initial value problem (3.3). Let $u : M \times [0, T) \to N$ be a map and set $\tilde{u} = i \circ u$. We want to show that if $(\tilde{u}) : M \times [0, T) \to i(N)$ is solution to the initial value problem (3.6), then u is a solution to the initial value problem (3.3). Hence from the footnote on the first part of the proof, we get

$$\Delta \tilde{u} = \operatorname{trace} \nabla di(du, du) + di(\tau(u)), \tag{3.28}$$

$$\operatorname{trace} \nabla di = \operatorname{trace} \nabla d\pi(di, di) + d\pi(\operatorname{trace} \nabla di) \quad \text{since } i = \pi \circ i. \tag{3.29}$$

Since $i : N \to \mathbb{R}^q$ is an isometric embedding, noting that $\operatorname{trace} \nabla di \in \Gamma(i^-1T\mathbb{R}^q)$ is orthogonal to $i(N)$ at each point, we get $d\pi(\operatorname{trace} \nabla di) = 0$. Therefore, we get $\operatorname{trace} \nabla di = \operatorname{trace} \nabla d\pi$. Substituting this in (3.28), we get

$$di(\tau(u)) = \Delta \tilde{u} - \operatorname{trace} \nabla d\pi(d\tilde{u}, d\tilde{u}) = \Delta \tilde{u} - \Pi(d\tilde{u}, d\tilde{u}) \quad \text{from (3.10)}. \tag{3.30}$$

On the other hand, since $di\left(\frac{\partial u}{\partial t}\right) = \frac{\partial \tilde{u}}{\partial t}$, and assuming (3.6) holds for $\tilde{u}$, we have

$$di\left(\tau(u) - \frac{\partial u}{\partial t}\right) = \left(\Delta - \frac{\partial}{\partial t}\right)\tilde{u} - \Pi(\tilde{u}, \tilde{u}) = 0. \tag{3.31}$$

Therefore, u is a solution of the initial value problem (3.3) as required.

The converse is clear from (3.30). □

From this, we see that we can get a time-dependent local solution to the initial value problem (3.3) by constructing a time-dependent local solution to the initial value problem (3.6). Since the new system of equations (3.6) is a system of parabolic differential equations with regard to the (Euclidean) vector valued functions, it is relatively easy to discuss the existence of solutions to this problem.

Theorem 3.1.3. *Let (M, g) and (N, h) be compact Riemannian manifolds. For any $C^{2+\alpha}$ map $f \in C^{2+\alpha}(M, N)$, there exists a positive number $\epsilon = \epsilon(M, N, f, \alpha) > 0$ and $u \in C^{2+\alpha, 1+\alpha/2}(M \times [0, \epsilon], \tilde{N})$ such that u is a solution in $M \times [0, \epsilon)$ to the initial value problem* (3.6).

Proof. Step 1 (Construction of an approximate solution). We linearlize the problem and construct an approximate solution. First by identifying f with $i \circ f$, we consider the following initial value problem.

$$\begin{cases} (\Delta - \frac{\partial}{\partial t})v(x,t) = \Pi(f)(df,df)(x), \quad (x,t) \in M \times (0,1), \\ u(x,0) = f(x). \end{cases} \tag{3.32}$$

Note that in the first equation, we have $\Pi(f)$ instead of $\Pi(v)$. This is a linear system of equations. By assumption on f, we get

$$f \in C^{2+\alpha}(M, \mathbb{R}^q), \quad \Pi(f)(df,df) \in C^{\alpha}(M, \mathbb{R}^q), \tag{3.33}$$

and therefore, there exists a unique solution $v \in C^{2+\alpha, 1+\alpha/2}(M \times [0,1], \mathbb{R}^q)$ to (3.32) from the theory of linear PDEs (Theorem A.2.2 in Appendix A.1). Denote the desired solution by u. Then v approximates u at $t = 0$ in the following sense.

$$v(x,0) = u(x,0), \quad \frac{\partial v(x,0)}{\partial t}|_{t=0} = \frac{\partial u(x,t)}{\partial t}|_{t=0}. \tag{3.34}$$

Step 2 (Application of the inverse function theorem). Set $Q = M \times [0,1]$ and consider a differential operator

$$P(u) = \Delta u - \frac{\partial}{\partial t} u - \Pi(u)(du,du). \tag{3.35}$$

A map $u \in C^{2+\alpha, 1+\alpha/2}(M \times [0,\epsilon], \mathbb{R}^q)$ satisfying $P(u) = 0$ is the desired solution.

Now for $0 < \alpha' < 1$, we define the subspaces X and Y in $C^{2+\alpha,1+\alpha/2}(Q,\mathbb{R}^q)$ and $C^{\alpha',\alpha'/2}(Q,\mathbb{R}^q)$, respectively, as follows:

$$X = \{z \in C^{2+\alpha',1+\alpha'/2(Q,\mathbb{R}^q)} | z(x,0) = 0, \frac{\partial z(x,t)}{\partial t} = 0\} \tag{3.36}$$

$$Y = \{k \in C^{\alpha',\alpha'/2(Q,\mathbb{R}^q)}(Q,\mathbb{R}^q) | k(x,0) = 0\}. \tag{3.37}$$

From the definitions, X and Y are closed subspaces of Banach spaces $C^{2+\alpha',1+\alpha'/2}$ and $C^{\alpha',\alpha'/2}$ and therefore are Banach spaces. For a given $z \in X$, if we put

$$\mathcal{P}(z) = P(v+z) - P(v), \tag{3.38}$$

where P is as defined above, then $\mathcal{P}$ defines a map $\mathcal{P} : X \to Y$ (since $z \in C^{2+\alpha,1+\alpha_2}(Q,\mathbb{R}^q)$, $\mathcal{P}(z) \in C^{\alpha,\alpha_2}(Q,\mathbb{R}^q)$). In particular $\mathcal{P}(0) = 0$. $\mathcal{P}$ is Fréchet differentiable in a neighborhood of $z = 0$ with the Fréchet derivative at $z = 0$ given by

$$\mathcal{P}'(0)(Z) = \Delta Z - \frac{\partial}{\partial t} Z - \sum_{A=1}^{q} Z^A \frac{\partial}{\partial z^A} \Pi(v)(dv, dv) - 2\Pi(v)(dv, dZ). \tag{3.39}$$

From this, we can see that $\mathcal{P}'(0)$ is an isomorphism. In fact, since $v \in C^{2+\alpha',1+\alpha'/2}(Q,\mathbb{R}^q)$, we see that for any $K \in Y$, there exists a unique $Z \in C^{2+\alpha',1+\alpha'/2}(Q,\mathbb{R}^q)$ such that

$$\begin{cases} (\mathcal{P}'(0))(Z)(x,t) = K(x,t), \quad (x,t) \in M \times (0,1), \\ Z(x,0) = 0. \end{cases} \tag{3.40}$$

with $|Z|_Q^{(2+\alpha',1+\alpha'/2)} \leq C|K|_Q^{\alpha',\alpha'/2}$ from the theory of PDEs (Theorem A.2.2 in Appendix A.1. Since $K(x,0) = 0$ and $Z(x,0) = 0$ holds, we have that $\frac{\partial}{\partial t} Z = 0$

holds; consequently, $Z \in X$. This tells us that $\mathcal{P}'(0)$ is surjective. Also $\mathcal{P}'(0)$ has a continuous inverse map as argued. Hence $\mathcal{P}'(0)$ is an isomorphism.

Applying inverse function theorem for Banach spaces[2] , $\mathcal{P} : X \to Y$ is a homeomorphism between sufficiently small neighborhood $\mathcal{U}$ of $0 \in X$ and a neighborhood $\mathcal{P}(\mathcal{U})$ of $0 \in Y$. In other words, there exists a positive number $\delta = \delta(M, N, f) > 0$ such that there exists a unique $z \in C^{2+\alpha',1+\alpha'/2}(Q, \mathbb{R}^q)$ satisfying the following conditions for any $k \in C^{\alpha',\alpha'/2}$ with $k(x, 0) = 0$ and $|k|_Q^{\alpha',\alpha'/2} < \delta$, z satisfies

$$\mathcal{P}(z) = k, \quad z(x, 0) = 0, \frac{\partial z(x,t)}{\partial t}\Big|_{t=0} = 0. \tag{3.41}$$

Here $\delta = \delta(M, N, f)$. Now if we set $u = v + z$ and $w = P(v)$, from (3.41), we see that there exists a $u \in C^{2+\alpha',1+\alpha'/2}(Q, \mathbb{R}^q)$ satisfying

$$\begin{cases} P(u)(x,t) = (w + k)(x,t), & (x,t) \in M \times (0,1) \\ u(x, 0) = f(x). \end{cases} \tag{3.42}$$

Step 3 (Existence of a time-dependent local solution) In order to prove the existence of the desired time dependent solution, for a given positive number ϵ, consider a C^∞ function $\zeta : \mathbb{R} \to \mathbb{R}$ satisfying $\zeta(t) = 1\,(t \leq \epsilon)$,$\zeta(t) = 0\,(t \geq 2\epsilon)$, $0 \leq \zeta(t) \leq 1$, $|\zeta'(t)| \leq 2/\epsilon\,(t \in \mathbb{R})$. We note that $w = P(v) \in C^{\alpha,\alpha/2}(Q, \mathbb{R}^q)$ and that $w(x, 0) = 0$

[2](Inverse function theorem for Banach spaces): Let $\mathcal{V}$ and $\mathcal{W}$ be Banach spaces. Given an open neighborhood U of $0 \in \mathcal{V}$, let $f : U \to \mathcal{W}$ be a map from U into $\mathcal{W}$ satisfying the following conditions:

i $f(0) = 0$ and f is Fréchet differentiable in U.

ii f is C^1 in Fréchet sense;

iii $f'(0) : \mathcal{V} \to \mathcal{W}$ is a homeomorphism; namely, $f'(0)$ is bijective and $f'(0)$ and its inverse map $f'(0)^{-1}$ are both bounded linear operators

Then there exists an open neighborhood $V \subset U$ of $0 \in U$ such that f is a homeomorphism from V onto an open neighborhood $f(V)$ of $0 \in \mathcal{W}$.

holds from the definition of $P(v)$ and $v(x,0) = f$. By computation, we can show that there is a constant $C > 0$, independent of ϵ and w such that

$$|\zeta w|_Q^{(\alpha',\alpha'/2)} \leq C\epsilon^{(\alpha-\alpha')/2}|w|_Q^{(\alpha,\alpha/2)} \tag{3.43}$$

holds.

Set $k = -\zeta w$. Then $K(x,0) = 0$. From (3.43), we have $|k|_Q^{2+\alpha',1+\alpha'/2} < \delta$ for ϵ small enough. Consequently, there exists a $u \in C^{2+\alpha',1+\alpha'/2}(M \times [0,\epsilon], \mathbb{R}^q)$ such that the following special case of (3.42) holds:

$$\begin{cases} P(u)(x,t) = 0, \quad (x,t) \in M \times (0,\epsilon) \\ u(x,0) = f(x). \end{cases} \tag{3.44}$$

Namely, we have obtained a solution $u \in C^{2+\alpha',1+\alpha'/2}(M \times [0,\epsilon], \mathbb{R}^q)$ to the initial value problem

$$\begin{cases} (\Delta - \frac{\partial}{\partial})u(x,t) = \Pi(u)(du,du), \quad (x,t) \in M \times (0,\epsilon) \\ u(x,0) = f(x). \end{cases} \tag{3.45}$$

Since we have

$$f \in C^{2+\alpha}(M, \mathbb{R}^q), \quad \Pi(u)(du,du) \in C^{\alpha,\alpha/2}(M \times [0,\epsilon], \mathbb{R}^q), \tag{3.46}$$

we see from the theory of PDEs (Theorem A.2.2 in Appendix A.1) that $u \in C^{2+\alpha,1+\alpha/2}(M \times [0,\epsilon], \mathbb{R}^q)$. Since $u(M \times [0,\epsilon']) \subset \tilde{N}$ for sufficiently small ϵ' such that $0 < \epsilon' < \epsilon$, u is a solution to the initial value problem (3.6) in $M \times [0,\epsilon']$. Applying Proposition 3.1.2, we see that u is a solution to (3.6) in $M \times [0,\epsilon]$. Clearly, ϵ is a positive number depending on M, N, f and α alone. $\square$

From the above theorem and Proposition 3.1.2, we get

Corollary 3.1.1. *Let (M,g) and (N,h) be compact Riemannian manifolds. For a given $C^{2+\alpha}$ map $f \in C^{2+\alpha}(M,N)$, there exist a positive number $T = T(M,N,f,\alpha) > 0$ and $u \in C^{2+\alpha,1+\alpha/2}(M \times [0,T], N)$ such that*

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \tau(u(x,t)), \quad (x,t) \in M \times (0,T), \\ u(x,0) = f(x) \end{cases} \tag{3.47}$$

holds. Here, $T = T(M,N,f,\alpha)$ is a constant dependent on M,N,f,α alone.

From the results given in Appendix A.1 regarding differentiability on the local solutions to a linear parabolic differential equation, we obtain

Theorem 3.1.4 (Existence of time-dependent local solutions)**.** *Let (M,g) and (N,h) be compact Riemannian manifolds. For a given $C^{2+\alpha}$ map $f \in C^{2+\alpha}(M,N)$, there exists a positive number $T = T(M,N,f,\alpha) > 0$ and $u \in C^{2+\alpha,1+\alpha/2}(M \times [0,T], N) \cap C^{\infty}(M \times (0,T), N)$ such that*

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \tau(u(x,t)), \quad (x,t) \in M \times (0,T), \\ u(x,0) = f(x) \end{cases} \tag{3.48}$$

holds. Here $T = T(M,N,f,\alpha)$ is a constant dependent on M,N,f and α alone.

Proof. Due to the Corollary 3.1.1, we only need to verify the differentiability of u at each point $(x,t) \in M \times (0,T)$. In the local coordinates as before, we express the parabolic equation for harmonic maps as

$$\left(\Delta - \frac{\partial}{\partial t}\right) u^{\alpha} = -\sum_{i,j=1}^{m} \sum_{\beta,\gamma=1}^{n} g^{ij} \Gamma'^{\alpha}_{\beta\gamma}(u) \frac{\partial u^{\beta}}{\partial x^i} \frac{\partial u^{\gamma}}{\partial x^j}. \tag{3.49}$$

From the previous corollary about u, the right hand side is $C^{1+\alpha,\alpha/2}$. From the theorem regarding differentiability of solutions to linear parabolic PDEs (Theorem A.2.2 in Appendix A.1), we get that u is $C^{3+\alpha,1+\alpha/2}$. This makes right hand side $C^{2+\alpha,1+\alpha/2}$ yielding u to be $C^{4+\alpha,1+\alpha/2}$ Repeating this argument, we get the desired result. $\square$

3.2 Existence of Global Time-Dependent Solutions

To show the existence of the global solution, we want to show that the initial value problem of the parabolic equation of the harmonic maps

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \tau(u(x,t)), \quad (x,t) \in M \times (0,T), \\ u(x,0) = f(x) \end{cases} \tag{3.50}$$

has a solution $u : M \times [0,\infty) \to N$ when $T = \infty$. Such a solution is called time-dependent global solution. As we saw in the previous section, time dependent local solution always exists. However, (3.50) is a system of non-linear equations; hence existence of a global solution is not guaranteed. We need to estimate *growth rate* of the solution $u(x,t)$ in time t. Here, the curvature of N plays a crucial role. We assume throughout that M and N are compact. Following lemma is a useful tool.

Lemma 3.2.1 (Maximum principle)**.** *Let $u \in C^0(M \times [0,T)) \cap C^{2,1}(M \times (0,T))$ be a real valued function in $M \times [0,T)$. If u satisfies $(\Delta - \partial/\partial t)u \geq 0$ in $M \times (0,T)$, then*

$$\max_{M\times[0,T]} u = \max_{M\times\{0\}} u \tag{3.51}$$

holds; namely the maximum value of u in $M \times [0,T)$ is attained at a point in $M \times \{0\}$.

Proof. Let $\epsilon_1, \epsilon 2 > 0$ be positive, and set

$$\hat{u}(x,t) = u(x,t) - \epsilon_1 t, \quad Q = M \times [0, T - \epsilon_2]. \tag{3.52}$$

Since $\hat{u}$ is a continuous function in Q, it attains the maximum value at point (x^o, y^o) in Q due to compactness. We claim that $t^o = 0$. Assume that $t^o > 0$ and we try to get a contradiction. Since $(\Delta - \partial/\partial t)u \geq 0$ in $M \times (0,T)$, from the assumption, $\hat{u}$ satisfies

$$\frac{\partial \hat{u}}{\partial t} \leq \Delta \hat{u} - \epsilon_1 \quad \forall (x,t) \in Q \tag{3.53}$$

in particular at (x^o, t^o). Namely, in local coordinates (x^i) about x^o,

$$\frac{\partial \hat{u}}{\partial t} \leq \sum_{i,j=1}^{m} g^{ij} \left\{ \frac{\partial^2 \hat{u}}{\partial x^i \partial x^j} - \sum_{k=1}^{m} \frac{\partial \hat{u}}{\partial x^k} \right\} - \epsilon_1 \tag{3.54}$$

holds at (x^o, y^o). But by assumption, (x^o, y^o) is maxima, and hence

$$\frac{\partial \hat{u}}{\partial t}(x^o, t^o) = 0, \quad \frac{\partial \hat{u}}{\partial x^i}(x^o, t^o) = 0 \tag{3.55}$$

and the matrix $\frac{\partial^2 \hat{u}}{\partial x^i \partial x^j}(x^o, y^o)$ is nonpositive definite. Substituting this in (3.54), we see that $\epsilon \leq 0$ which contradicts the assumption that it is positive. This proves the assertion that $t^o = 0$ and the assertion is true for $\hat{u}$. Noting that ϵ_1 and ϵ_2 were arbitrary, we obtain the desired result. $\square$

Next we use Weizenböck formula to get the following estimate on rate of growth of u.

Proposition 3.2.2. *Let* $u \in C^{2,1}(M \times [0,T), N) \cap C^\infty(M \times (0,T), N)$ *be a solution to* (3.50) *and set* $u_t(x) = u(x,t)$*. Assume that* N *has nonpositive curvature* $K_N \leq 0$ *and*

$Ric^M \geq -Cg$ for a constant $C \in R$. Furthermore, let ϵ be a positive number such that $0 < \epsilon < T$. Then the following holds for the energy density functional $e(u)$ of u:

1. *For an arbitrary $(x,t) \in M \times (0,T)$,*

$$e(u_t)(x) \leq e^{2Ct} \sup_{x \in M} e(f)(x). \tag{3.56}$$

2. *For an arbitrary $(x,t) \in M \times [\epsilon, T)$,*

$$e(u_t)(x) \leq C(M, \epsilon) E(f). \tag{3.57}$$

Here $C(M,t)$ is a constant dependent on only M and ϵ.

Proof. (1) From Corollary 2.2.2, we get

$$\left(\Delta - \frac{\partial}{\partial t}\right) e(u_t) \geq -2Ce(u_t) \tag{3.58}$$

If we put $v(x,t) = e^{-2Ct}e(u_t)$, we see that from the above inequality, v satisfies $(\Delta - \partial/\partial t)v \geq 0$ in $M \times (0,T)$. hence from the maximum principle proved in Lemma 3.2.1,

$$e^{-2Ct}e(u_t)(x) = v(x,t) \leq \max_{x \in M} v(x,0) = max_{x \in M} e(f)(x) \tag{3.59}$$

holds for arbitrary $(x,t) \in M \times [0,T)$. (2)Similarly computation. □

The above proposition puts bounds on the rate at which the energy density functional decreases. Another important bound on rate of growth of the solution is given in the following proposition.

Proposition 3.2.3. *Let* $u \in C^{2,1}(M \times [0,T), N) \cap C^{\infty}(M \times (0,T), N)$ *be a solution to* (3.32). *If* N *is of nonpositive curvature* $K_N \leq 0$*, at any* (x,t)*, we have*

$$\left|\frac{\partial u}{\partial t}(x,t)\right| \leq \sup_{x \in M} \left|\frac{\partial u}{\partial t}(x,0)\right|. \tag{3.60}$$

Proof. From corollary to the Weizenböck inequality Corollary 2.2.2, we have

$$\left(\Delta - \frac{\partial}{\partial t}\right) \kappa(u_t) \geq 0 \tag{3.61}$$

for $u_t(x) = u(x,t)$. Since by definition $\kappa(u_t) = \frac{1}{2}\left|\frac{\partial u(x,t)}{\partial t}\right|^2$, we get the desired result from the maximum principle Lemma 3.2.1. □

From Propositions 3.2.2 and 3.2.3, we can get that the growth rate of a solution u to the initial value problem (3.32) is uniformly bounded with respect to time, if K_N curvature of N is nonpositive.

Theorem 3.2.4 (Existence of time-dependent global solutions)**.** *Let* (M,g) *and* (N,h) *be compact Riemannian manifolds, and assume that* N *is of non-positive curvature* $K_N \leq 0$*. Then for any* $C^{2+\alpha}$ *map* $f \in C^{2+\alpha}(M,N)$*, there exists a unique* $u \in C^{2+\alpha, 1+\alpha/2}(M \times [0,\infty), N) \cap C^{\infty}(M \times (0,\infty), N)$ *such that*

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \tau(u(x,t)), \quad (x,t) \in M \times (0,\infty), \\ u(x,0) = f(x) \end{cases} \tag{3.62}$$

holds.

Proof. We have seen in Theorem 3.1.4 that a time-dependent local solution to problem (3.62) exists. Namely, there exists a positive number $T = T(M,N,f,\alpha) > 0$ such that, regardless of the curvature of N, the initial value problem (3.62) has a solution

$u \in C^{2+\alpha,1+\alpha/2}(M \times [0,\infty), N) \cap C^{\infty}(M \times (0,\infty), N)$. Now we claim that if N is of nonpositive curvature $K_N \leq 0$, then this solution u can be extended to $M \times [0,\infty)$. Let

$$T_0 = \sup\{t \in [0,\infty) \,|\, \text{(3.62) has a solution}\}. \tag{3.63}$$

We will show that $T_0 = \infty$. Assume $T_0 < \infty$, and let t_i be a sequence converging to T_0. By the Nash embedding theorem, we can consider N to be a submanifold of some Euclidean space R^q. Therefore, we can regard each $u(\cdot, t_i) \in C^{\infty}(M, N)$ as a vector valued function $u : M \to \mathbb{R}^q$. We see that for $0 < \alpha < \alpha' < 1$, the sequence of functions $\{u(\cdot, t_i)\}$ and $\{\frac{\partial u(\cdot,t)}{\partial t}\}$ form uniformly bounded subsets in the function spaces $C^{2+\alpha'}$ and $C^{\alpha'}(M, \mathbb{R}^q)$, respectively. Hence, these sequences become, respectively, uniformly bounded and continuous subsets in the function spaces $C^{2+\alpha}(M, \mathbb{R}^q)$ and $C^{\alpha}(M, \mathbb{R}^q)$. By the Ascoli-Arzelà theorem, there exists a subsequence $\{t_{i_k}\}$ of $\{t_i\}$ and functions

$$u(\cdot, T_0) \in C^{2+\alpha}(M, \mathbb{R}^q) \text{ and } \frac{\partial u(\cdot, T_0)}{\partial t} \in C^{\alpha}(M, \mathbb{R}^q) \tag{3.64}$$

such that the subsequences $\{u(\cdot, t_{i_k})\}$ and $\{\frac{\partial u(\cdot,t_{i_k})}{\partial t})\}$ respectively, converge to $u(\cdot, T_0)$ and $\frac{\partial u(\cdot,T_0)}{\partial t}$, as $t_{i_k} \to T_0$. Since we have $\frac{\partial u(\cdot,t_{i_k})}{\partial t} = \tau(u(\cdot, t_(i_k)))$. Therefore due to uniform convergence, we also get at T_0, $\frac{\partial u(\cdot,T_0)}{\partial t} = \tau(u(\cdot, T_0))$. Thus we see that (3.62) has a solution in $M \times [0, T_0]$. By applying Theorem 3.1.4 on local time-dependent solution, with $u(\cdot, T_0)$ as an initial value, we can extend the solution $u \in [0, T_0]$ to $u \in [0, T_0 + \epsilon)$ for some $\epsilon > 0$. This contradicts the definition of T_0. Thus $T_0 = \infty$. □

This concludes the proof of global existence and hence proves the Eells Sampson theorem.

Bibliography

[Do 76] M. Do Carmo. *Differential Geometry of Curves and Surfaces*. Prentice-Hall, 1976.

[ES64a] J. Eells and J. H. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.*, 86:109–160, 1964.

[ES64b] J. Eells and J. H. Sampson. Harmonic mappings of Riemannian manifolds. *Ann. J. Math.*, pages 109–160, 1964.

[Eva98] Evans. *Partial Differential Equations*. AMS, 1998.

[EW76] J. Eells and J. C. Wood. Restrictions of harmonic maps. *Topology*, 15:263–266, 1976.

[Fol84] G. Folland. *Real Analysis*. Wiley, New York, 1984.

[Ham75] R. Hamilton. Harmonic maps of manifolds with boundary. In *Lecture Notes in Mathematics*, number 471. Springer, 1975.

[Jos02] J. Jost. *Riemannian Geometry and Geometric Analysis*. Springer Verlag, 2nd edition, 2002.

[Nas56] J. Nash. The embedding problem for Riemannian manifolds. *Ann. Math.*, 63, 1956.

[Nis01] Seiki Nishikawa. *Variational Problems in Geometry*, volume 205 of *Translations of Mathematical Monographs*. AMS, 2001.

[Roy88] H. L. Royden. *Real Analysis*. Macmillan Publishing Company, New York, 3rd edition, 1988.

[Sam78] J. H. Sampson. Some properties and applications of harmonic mappings. *Ann. Sc. Ec.*, Supp. 11:211–228, 1978.

[SU82] R. Schoen and K. Uhlenbeck. A regularity theory for harmonic maps. *J. Diff. Geom.*, 17:305–335, 1982.

Appendix A

Hölder Spaces and Some Results on PDEs

In this chapter we state some of the results we used without proof.

A.1 Hölder Spaces

Given $T > 0$, let $Q = M \times [0, T]$. Let $0 < \alpha < 1$. Given a vector valued function $u : Q \to \mathbb{R}^q$, set

$$|u|_Q = \sup_{(x,t)\in Q} |u(x,t)|, \tag{A.1}$$

$$\langle u\rangle_x^{(\alpha)} = \sup_{(x,t),(x',t)\in Q\ x\neq x'} \frac{|u(x,t)-u(x',t)|}{d(x,x')^\alpha} \tag{A.2}$$

$$\langle u\rangle_t^{(\alpha)} = \sup_{(x,t),(x,t')\in Q\ t\neq t'} \frac{|u(x,t)-u(x,t')|}{|t-t'|^\alpha} \tag{A.3}$$

where $d(x, x')$ is the distance between x and x' in M. Using these norms, we define the norms $|u|_Q^{(\alpha,\alpha/2)}$, $|u|_Q^{(2+\alpha,1+\alpha/2)}$ by

$$|u|_Q^{(\alpha,\alpha/2)} = |u|_Q + \langle u\rangle_x^{(\alpha)} + \langle u\rangle_x^{(\alpha/2)}, \tag{A.4}$$

$$|u|^{(2+\alpha,1+\alpha/2)} = |u|_Q + |\frac{\partial u}{\partial t}|_Q + |D_x u|_Q + |D_x^2 u|_Q + \langle\frac{\partial u}{\partial t}\rangle_t^{(\alpha/2)} \tag{A.5}$$

$$+ \langle D_x u\rangle_t^{(1/2+\alpha/2)} + \langle D_x^2 u\rangle_t^{(\alpha/2)} \tag{A.6}$$

$$+ \langle\frac{\partial u}{\partial t}\rangle_x^{(\alpha)} + \langle D_x^2 u\rangle_x^{(\alpha)}. \tag{A.7}$$

Here D_x and D_x^2 denote the first order derivative and its covariant derivative, namely

$$D_x u = du = \sum_{i=1}^{m} \sum_{\alpha=1}^{q} \frac{\partial u^\alpha}{\partial x^i} \cdot dx^i \otimes \frac{\partial}{\partial y^\alpha}, \tag{A.8}$$

$$D_x^2 u = \nabla du = \sum_{i,j=1}^{m} \sum_{\alpha=1}^{q} \nabla_i \nabla_j u^\alpha \cdot dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^\alpha}. \tag{A.9}$$

Norm on these is defines by,

$$|D_x u|_Q^2 = \sup_{(x,t)\in M} \sum_{i,j=1}^{m} \sum_{\alpha=1}^{q} g^{ij} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\alpha}{\partial x^j}, \tag{A.10}$$

$$|D_x^2 u|_Q^2 = \sup_{(x,t)\in M} \sum_{i,j,k,l=1}^{m} \sum_{\alpha=1}^{q} g^{ik} g^{jl} \nabla_i \nabla_j u^\alpha \nabla_k \nabla_l u^\alpha. \tag{A.11}$$

Given non-negative integers κ and α, the set of all C^κ continuous functions $u : Q \to \mathbb{R}$ whose κ-th partial derivatives are α-Hölder-continuous is denoted by $C^{\kappa+\alpha}(\bar{Q})$ and is called *Hölder space*. The Hölder space $C^{\kappa+\alpha}(\bar{Q})$ becomes a Banach space under the norm

$$|u|^{\kappa+\alpha} = \sum_{|\beta|\le\kappa} \sup_Q |D^\beta u| + \sum_{|\beta|=\kappa} \langle D^\beta u \rangle_Q^\alpha. \tag{A.12}$$

where β denote multi-index. Set

$$C^{\kappa+\alpha}(Q) = \{u \in C^\kappa(Q) | \ |u|_Q^{\kappa+\alpha} < \infty\}.$$

With respect to these norms and function spaces, we define the function spaces $C^{\alpha,\alpha/2}(Q,\mathbb{R}^q)$ and $C^{2+\alpha,1+\alpha/2}(Q,\mathbb{R}^q)$, respectively, by

$$C^{\alpha,\alpha/2}(Q,\mathbb{R}^q) = \{u \in C^0(M\times[0,T]) |\ |u|_Q^{(\alpha,\alpha/2)} < \infty\}, \tag{A.13}$$

$$C^{2+\alpha,1+\alpha/2}(Q,\mathbb{R}^q) = \{u \in C^{2,1}(M\times[0,T]) |\ |u|_Q^{(2+\alpha,1+\alpha/2)} < \infty\}, \tag{A.14}$$

$$C^{2+\alpha,1+\alpha/2}(Q,N) = \{u \in C^{2+\alpha,1+\alpha/2}(M\times[0,T]) |\ u(Q)\subset N\}. \tag{A.15}$$

$C^{\alpha,\alpha/2}(Q,\mathbb{R}^q)$ and $C^{2+\alpha,1+\alpha/2}(Q,\mathbb{R}^q)$ are Banach spaces with norms $|u|_Q^{\alpha,\alpha/2}$, $|u|_Q^{2+\alpha,1+\alpha/2}$, respectively. Also it can be shown that $C^{2+\alpha,1+\alpha/2}(Q,N)$ is a closed subset of $C^{2+\alpha,1+\alpha/2}(Q,\mathbb{R}^q)$. $C^{\alpha,\alpha/2}(Q,\mathbb{R}^q)$ and $C^{2+\alpha,1+\alpha/2}(Q,\mathbb{R}^q)$ are called a *Hölder space* on $Q = M\times[0,T]$.

A.2 Some Results on PDEs

Let $\Omega \in \mathbb{R}^m$ be a bounded and connected open set, and let P be a linear elliptic partial differential operator given by

$$P = \sum_{i,j=1}^{m} a^{ij}(x)\frac{\partial^2}{\partial x^i\partial x^j} + \sum_{i=1}^{m} b^i(x)\frac{\partial}{\partial x^i} + d(x). \tag{A.16}$$

Theorem A.2.1. *(1) Given $0<\alpha<1$, assume that $a^{ij}, b^i, d, f \in C^\alpha(\Omega)$. Then $u \in C^{2+\alpha}(\Omega)$ if $u \in C^2(\Omega)$ satisfies a linear elliptic partial differential equation $Pu(x) = f(x)$. (2) Furthermore, if $a^{ij}, b^i, d, f \in C^{\kappa+\alpha}(\Omega)$ for given $\kappa \geq 1$, then the solution u to (1) is $C^{\kappa+\alpha+2}$. In particular if $a^{ij}, b^i, d, f \in C^\infty(\Omega)$, then $u \in C^\infty(M)$.*

Theorem A.2.2. *Let (M, g) be a compact Riemannian manifold, and set $Q = M \times [0, T]$. Given a vector valued function $u : Q \to \mathbb{R}^q$, let*

$$Lu = \Delta u + \vec{a} \cdot \nabla u + \vec{b} \cdot u - \frac{\partial}{\partial t} u \tag{A.17}$$

be a parabolic partial differential operator, and consider an initial value problem

$$\begin{cases} Lu(x,t) = F(x,t), \quad (x,t) \in M \times (0,T) \\ u(x,0) = f(x). \end{cases} \tag{A.18}$$

Here, the components of $\Delta u, \vec{a} \cdot \nabla u, \vec{b} \cdot u, \frac{\partial}{\partial t} u$ are defined by

$$\Delta u^A, \qquad \sum_{B=1}^{q} \sum_{i=1}^{m} a_B^{iA}(x,t) \frac{\partial u^B}{\partial x^i}, \qquad \sum_{B=1}^{q} b_B^A(x,t) u^B, \qquad \frac{\partial u^B}{\partial t}. \tag{A.19}$$

If $a_B^{iA}, b_B^A \in C^{\alpha,\alpha/2}(Q, \mathbb{R}), \quad 1 \le m,\ 1 \le A, B \le q$, for some $0 < \alpha < 1$, then, for any

$$F \in C^{\alpha,\alpha/2}(Q, \mathbb{R}), \quad f \in C^{2+\alpha}(M, \mathbb{R}^q), \tag{A.20}$$

there exists a unique solution $u \in C^{2+\alpha,1+\alpha/2}(Q, \mathbb{R}^q)$ to (A.18) *such that*

$$|u|_Q^{(2+\alpha,1+\alpha/2)} \le C(|F|_Q^{(\alpha,\alpha/2)} + |f|_M^{(2+\alpha)}) \tag{A.21}$$

holds. Here $C = C(M, L, q, T, \alpha)$ is a constant dependent on M, L, q, T, α alone.

A.3 Fréchet Derivative

Definition A.3.1. Let X and Y be normed linear spaces, $U \in X$ open, $f : X \to Y$ and $x \in U$. Then we say f is ***Fréchet differentiable*** at x if there is a bounded linear $A \in B(X, Y)$ such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Ah\|_Y}{\|h\|_X} = 0. \tag{A.22}$$

We call A the *Fréchet derivative* of f at x and denote it by $f'(x) = Df(x)$.

Proposition A.3.1. *If f is Fréchet differentiable at x, then $f'(x) = Df(x)$ is unique and f is continuous at x.*